Research summary

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1 Introduction

I have a Ph.D. in mathematics from Purdue University, Indiana, USA. My Ph.D. advisor was Professor Ahmed Sameh. I worked on the parallel solution of narrow banded linear systems and on parallel algorithms for the numerical solution of Lyapunov matrix equations with applications in truncation based model reduction. Since July 1st, 2009 I am continuing my work as a post doctoral researcher at Professor Bo Kågström's group at the Department of Computing Science and HPC2N at Umeå University.

2 Narrow banded linear systems

Consider the standard linear system

$$Ax = f$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular, sparse matrix and $f \in \mathbb{R}^n$. Frequently, but not universally, it is possible to find a pair of permutation matrices P and Q, such that the matrix $A' = PAQ^T$ is either narrow banded or admits a good narrow banded preconditioner. In addition, narrow banded linear systems occur naturally in many different settings. The numerical solution of parabolic differential equations using compact difference methods is a good source of examples.

The truncated SPIKE algorithm applies to narrow banded linear systems which are strictly diagonally dominant by rows [15, 16]. The parallel bottleneck consists of the solution of relatively small block tridiagonal linear systems. Frequently, the off diagonal blocks are negligible and can be dropped, a phenomenon which greatly accelerates the linear solve. I have developed tight estimates of the absolute size of the off diagonal blocks [6]. This particular question is closely related to the problem of estimating the decay rate of the elements of the inverse of a narrow banded matrix. I have developed estimates which are tight for matrices which are strictly diagonally dominant by rows and I have characterized the set of tridiagonal matrices which exhibit the slowest possible decay rate [7].

Recently, I have shown that the same phenomenon can be exploited in the current ScaLAPACK solver [11]. The algorithm in question is block cyclic reduction corresponding to a particular partitioning of the banded matrix. It is well known that the significance of the off diagonal blocks decay quadratically with the number of cyclic reduction steps. However, I have just developed tight estimates for the significance of the off diagonal blocks in the initial Schur complement and I can characterize the worst case behavior. It is the results of these investigations which I would like present at the Householder Symposium 2011. This work represents an extension of D. Hellers analysis of cyclic reduction for block tridiagonal linear systems which are strictly diagonally dominant by rows [2].

3 Lyapunov matrix equations

Let A be an n by n matrix and let B be an n by p matrix and consider the Lyapunov matrix equation

$$AX + XA^T + BB^T = 0.$$

If A is stable, then this equation has a unique solution X, which is symmetric positive semidefinite, and $\operatorname{Ran} X$ is the smallest A invariant subspace containing the range of B.

In many applications B is a tall matrix with $O(1)$ columns. Frequently, the eigenvalues for X decay rapidly and X can be approximated accurately with a low rank matrix. This is the low rank phenomenon for Lyapunov matrix equations. During the last 20 years a number of methods have been developed in order to compute good low rank approximations to X directly. The low rank cyclic Smith method has been particularly successful, but it requires the selection of certain shift parameters and the proper choice can be exceedingly difficult [12, 14]. The search for parameter free methods is therefore fully justified.

It is well known that the Lyapunov matrix equation is equivalent to a standard linear system of dimension n^2. Specifically,

$$(A \otimes I + I \otimes A)\operatorname{vec}(X) + \operatorname{vec}(BB^T) = 0,$$

where $\otimes$ denotes the Kronecker product and vec is the operator, which stacks the columns of X on top of each other to form a single vector $\operatorname{vec}(X) \in \mathbb{R}^{n^2}$.

Obviously, it is possible to apply Krylov subspace methods directly to Lyapunov matrix equations in Kronecker product form. However, the naive implementation would require $O(n^2)$ arithmetic operations and $O(n^2)$ words of storage. I have shown that it is possible to reduce these demands to $O(n)$, by exploiting a very compact representation of certain vectors in $\mathbb{R}^{n^2}$ [8]. Unfortunately, the representation does not permit preconditioning in the usual sense, and the convergence can be very slow.

The standard Arnoldi method for Lyapunov matrix equations remains a popular choice [13, 4]. However, it is well known that the convergence can be very slow and at least one attempt has been made to reduce the storage requirements at the expense of an increase number of arithmetic operations [5]. I have given a constructive proof of the fact that any positive residual history is possible [9].

Recently, the extended Krylov subspace method has been applied to the solution of Lyapunov matrix equations [17]. In spirit, this method is similar to the standard Arnoldi method, but in practice it converges much more rapidly. Regardless, I have shown that any possible residual curve is possible [10].

I have extended Hodel's work on the approximate power iteration for Lyapunov matrix equations [3]. Specifically, I have shown that it is possible to compute the dominant subspaces for X using an approximate subspace iteration, provided that A is negative definite and X has numerical low rank [8]. However, the algorithm requires the solution of tall Sylvester equations which are updated at each iteration. In addition, the theoretical analysis continues to elude me.

Surveying the field, I find it lacking a method which can obtain a good low rank approximation of the exact solution X, assuming only that A is stable and the singular values of X decay rapidly.

References

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