Image Factorization in Relative Categories

by

Stephen J. Hegner
Department of Computer Science and Electrical Engineering
Votey Building
University of Vermont
Burlington, VT 05405

July, 1982

ABSTRACT

The concept of image-factorization system is generalized from the ordinary category case to that of a category over a closed monoidal category. The main feature of the generalization is the recapturing of the lifting theorems to functor categories and categories of algebras. Conditions under which an ordinary image-factorization system is also a relative image-factorization system are also presented, together with detailed examples over the category of Banach spaces.

§0. INTRODUCTION

For some time now, it has been recognized that ordinary category theory, which is based upon the category **Set** of sets, is not a satisfactory tool in all applications. For this reason, the idea of a category relative to a closed monoidal category has been developed [8,15]. In this paper, we investigate the extension of an important categorical concept -- that of an image-factorization system -- to the relative case.

Image-factorization systems in ordinary categories are the natural generalization of the fact that every function $f : A \longrightarrow B$ has a unique (up to isomorphism) factorization $A \xrightarrow{e} C \xrightarrow{m} B$, with e a surjection and m an injection [1,2,3,13]. In further extending this notion to categories over a closed monoidal category other than set, it is first necessary to establish exactly what properties are essential to the extension. Our motivation comes from the categorical theory of continuous-time linear systems, and so we briefly outline our goals from the point of view of that framework.

In mathematical system theory, the application of image-factorization systems is to the so-called *realization problem*, in which an external (or input-output) description of the system is realized by a special internal dynamics -- called the canonical realization [1,2]. The process of realization typically involves the lifting of an image-factorization system from a base category (*e.g.* **Set** for discrete-time automata) to a category of dynamics, which is usually isomorphic to a functor category or a category of algebras (*e.g.* actions of a free monoid X^* in the case of discrete-time automata [2,4]). Similar lifting-type results apply in the case of discrete-time linear systems [1]. However, in work on the (ordinary) categorical theory of continuous-time linear systems, the lifting results completely lose their categorical flavor, and must be established on a case-by-case basis [10,11]. The underlying reason for this is that the dynamics for continuous-time linear systems, which are (C_o) semigroups of operators [11], are not describable as algebras of an algebraic theory over **Set**, as are discrete-time dynamics. Rather, they are represented as algebras of an algebraic theory over an appropriate closed monoidal category of locally convex spaces [12]. Our goal, then, is to generalize the concept of image-factorization system to relative categories in such a way that the well-known lifting theorems to functor categories and categories of algebras are appropriately generalized.

Guided by the motivation outlined in the above paragraph, we present a development of the concept of relative image-factorization system. The outline of the paper is as follows. Section 1 provides a quick review and establishes terminology for ordinary image-factorization systems. Section 2 places the concepts of image-factorization systems into a relative category-theory framework. Sections 3 and 4 provide support for the usefulness of the generalizations proposed in Section 2. In Section 3, we show that just as ordinary image-factorization systems lift to functor categories, so too do our relative image-factorization systems lift to relative functor categories. In Section 4, we provide similar lifting results to categories of relative algebras, generalizing the known results for ordinary categories of algebras. Section 5 provides results comparing our relative image-factorization systems to the ordinary variety. In Section 6, we provide detailed examples within the closed monoidal category of Banach spaces, including some ordinary image-factorization systems which are not image-factorization systems in the relative sense.

For example applications of the lifting results of Sections 3 and 4, we refer the reader to [12]. We have not included them here because of their highly specialized functional-analytic nature. Our emphasis here is on the purely categorical nature of the underlying results.

For consistency, wherever possible we have adhered to the general categorical terminology and notation of Herrlich and Strecker [13]. For relative concepts, the paper of Bunge [5] is our source.

§1. FUNDAMENTALS OF IMAGE-FACTORIZATION SYSTEMS

Although the idea of image factorization has been used by numerous researchers, the basic terminology and ideas are nowhere near standardized. This section serves the purposes of establishing the definitions we will be generalizing in this paper, as well as of restating some of the well-known results in a context more suitable for our purposes. Our basic terminology and definitions are those used by Arbib and Manes [1,2,3], although they differ in only inessential ways from those of Herrlich and Strecker [13].

1.1 DEFINITION Let **A** be a category. An *image-factorization system* (IFS) for **A** is a pair (**E**,**M**), where **E** and **M** are classes of **A** morphisms subject to the following axioms.

I1. (a) **E** is closed under composition.

(b) **M** is closed under composition.

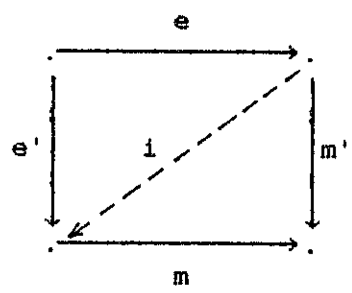
I2. (a) $e \in \mathbf{E} \implies$ e is an epimorphism.

(b) $m \in \mathbf{M} \implies$ m is a monomorphism.

I3. i is an isomorphism $\implies i \in \mathbf{E} \cap \mathbf{M}$.

I4. Every $f \in$ Morphisms(**A**) has a factorization $f = m \circ e$, with $e \in \mathbf{E}$ and $m \in \mathbf{M}$.

I5. If $m \circ e = m' \circ e'$ with $e, e' \in \mathbf{E}$ and $m, m' \in \mathbf{M}$, then there exists an isomorphism i such that the following diagram commutes.

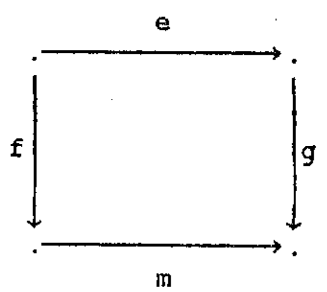


Given an IFS (**E**,**M**) and an **A**-morphism f, a pair (e,m) ∈ **E** × **M** with f = m∘e is called an (**E**,**M**)-factorization of f.

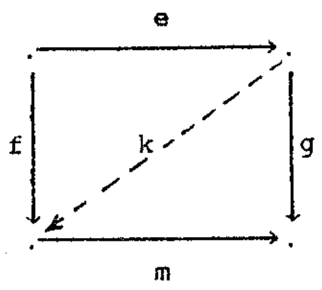
In generalizing the concept of IFS to the relative case, it is necesary to restate I5 in a way that does not draw diagrams in **A**, but rather expresses the property as a diagram in the base category (which is **Set** in the ordinary case). Theorem 1.3 provides the needed result, with the diagonal fill-in lemma as a preliminary step.

1.2 DIAGONAL FILL-IN LEMMA *In the definition of IFS, condition* I5 *may be replaced with the following axiom.*

I5'. For every commutative diagram in **A** of the form



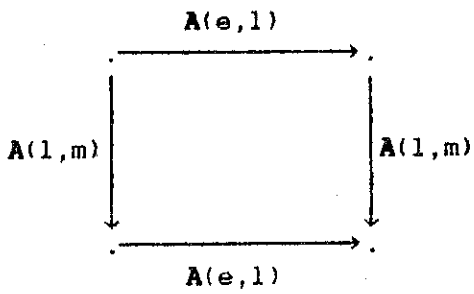
with (e,m) ∈ **E** × **M**, there is a unique k ∈ Morphisms(**A**) such that the following diagram commutes.



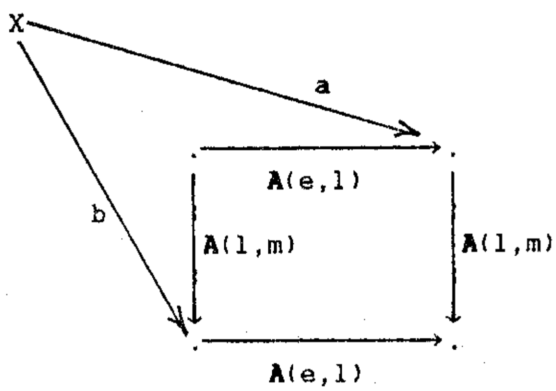
Proof: Consult [1, 3.5] or [13, Thm. 33.3]. ∎

1.3 THEOREM *The diagonal fill-in lemma condition* I5' *is equivalent to the following pullback condition.*

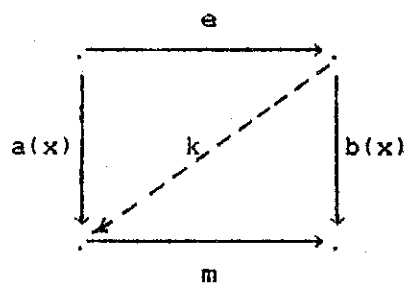
I5". For each pair $(e,m) \in \mathbf{E} \times \mathbf{M}$, the following diagram is a pullback in **Set**.



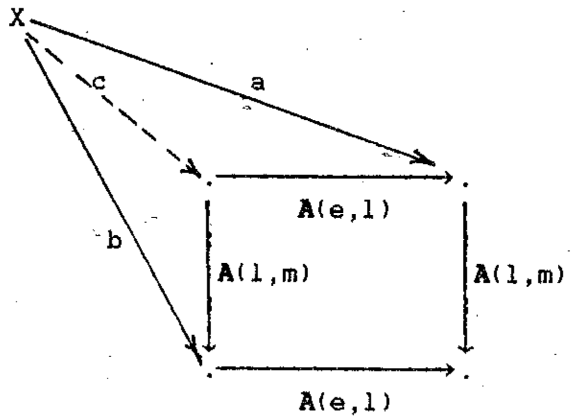
Proof: Assume that I5' holds. The above diagram surely commutes. Let X be any set, and suppose that a and b are functions such that



commutes. Define c on X by $c : x \mapsto k$, where k is the unique fill-in in the diagram below.

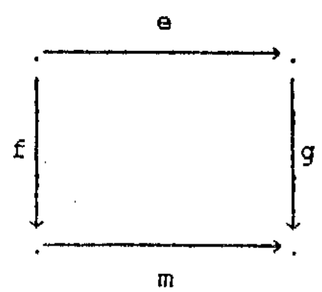


Clearly c is the unique fill-in

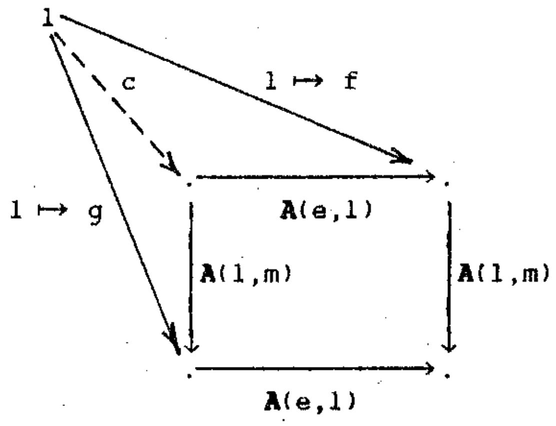


so we indeed have a pullback.

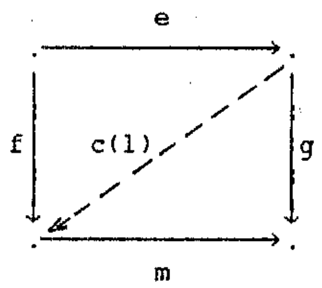
Conversely, suppose that I5" holds, suppose that f and g are **A**-morphisms, and further suppose that for each $(e,m) \in \mathbf{E} \times \mathbf{M}$, the following diagram commutes.



Construct the pullback



where 1 denotes the one-element set. Clearly c(1) provides the unique fill-in



so that I5' holds. ∎.

§2. RELATIVE CATEGORIES AND RELATIVE IMAGE-FACTORIZATION SYSTEMS

Ordinary category theory is based upon the category **Set** of sets. Relative category theory generalizes this by replacing **Set** with another category **V**. Although early work in the field placed as little structure on **V** as was absolutely necessary, and then added axioms as needed [8], it has now become recognized that the minimal structure for **V** is that of a symmetric closed monoidal category [5,7]. Accordingly, we adopt this structure as our starting point.

We present here only the notation and terminology necessary to clarify our presentation. See [5], [7], or [14] for details of relative category theory.

If **V** is a symmetric closed monoidal category (hereafter *closed category*), we let $-\otimes-$ denote the tensor product and Z denote the identity object, unless otherwise indicated. $\mathbf{V}_o$ denotes **V** when regarded as just an ordinary category, without any additional structure. If **A** is a **V**-category, $\mathbf{A}_o$ denotes its (ordinary) underlying category, with $\mathbf{A}_o(A,B) = \mathbf{V}_o(Z,\mathbf{A}(A,B))$, and we let **V** denote the **V**-category with underlying category $\mathbf{V}_o$ via the isomorphism $\mathbf{V}(A,B) \cong \mathbf{V}(A\otimes Z,B) \cong \mathbf{V}(Z,\mathbf{V}_o(A,B)) = \mathbf{V}_o(A,B)$.

Throughout the rest of this section, fix a closed category **V**. The following generalization of epimorphism and monomorphism to the **V** case is from [7].

2.1 DEFINITION Let **A** be a **V** category, and let $f : A \to B$ be a morphism in $\mathbf{A}_o$. f is a **V**-*monomorphism* (resp. **V**-*epimorphism*) if for every $C \in$ Objects(**A**), $\mathbf{A}(C,f) : \mathbf{A}(C,A) \to \mathbf{A}(C,B)$ (resp. $\mathbf{A}(f,C) : \mathbf{A}(B,C) \to \mathbf{A}(A,C)$) is a monomorphism in **V**.

2.2 OBSERVATION *Let* **A** *be a* **V**-*category. Then every* **V**-*monomorphism is a* *monomorphism, and every* **V***-epimorphism is an epimorphism.*

Proof: Hom functors preserve monomorphisms [13, Prop. 12.13]. ∎

We are now ready to present our proposed definition of an image-factorization system relative to **V**.

2.3 DEFINITION Let **A** be a **V**-category. A **V**-*image-factorization system* (**V**-*IFS*) for **A** is a pair (**E**,**M**), where **E** and **M** are classes of $\mathbf{A}_o$ morphisms subject to the following axioms.

R-I1. (a) **E** is closed under composition.

(b) **M** is closed under composition.

R-I2. (a) $e \in \mathbf{E} \implies e$ is a **V**-epimorphism.

(b) $m \in \mathbf{M} \implies m$ is a **V**-monomorphism.

R-I3. i is an isomorphism $\implies i \in \mathbf{E} \cap \mathbf{M}$.

R-I4. Every $f \in \text{Morphisms}(\mathbf{A}_o)$ has a factorization $f = m \circ e$ with $e \in \mathbf{E}$ and $m \in \mathbf{M}$.

R-I5. For every $(e,m) \in \mathbf{E} \times \mathbf{M}$, the following diagram is a pullback in **V**.

$$\begin{array}{ccc} \cdot & \xrightarrow{\mathbf{A}(e,1)} & \cdot \\ {\scriptstyle \mathbf{A}(1,m)}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(1,m)} \\ \cdot & \xrightarrow[\mathbf{A}(e,1)]{} & \cdot \end{array}$$

Note that only two changes have been made to the ordinary definition. First of all, I2 has been changed so that the epimorphisms and monomorphisms are relative to $\mathbf{V}_o$. As we shall see in section 5, I2 and R-I2 are equivalent in most examples of interest. The second, and more interesting change is in I5. We have essentially lifted I5" from **Set** to **V**. This represents the key to our enhanced defintion; it is the property which provides the tools to lift image-factorizations to relative functor categories and relative algebra categories. Conditions for ensuring R-I2 and R-I5 are presented in section 5. Finally, in section 6, we show by example that not every IFS is a **V**-IFS, so that the R- conditions are indeed stronger. Of course, every **V**-IFS is an ordinary IFS; we record this fact formally.

2.4 OBSERVATION *Every* **V**-*IFS is an (ordinary) IFS*.

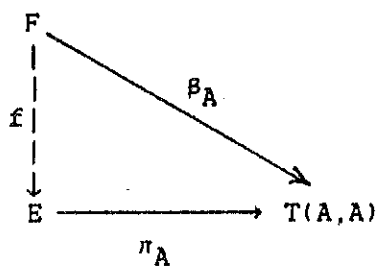
Proof: R-I2 follows immediately from Observation 2.2. R-I5 follows from the equivalence developed in Theorem 1.3. R-I1, R-I3, and R-I4 remain unchanged from their ordinary counterparts.∎

§3. LIFTING TO RELATIVE FUNCTOR CATEGORIES

As before, we let **V** be a closed category. We also fix **V**-categories **A** and **B** with **A** small. $[\mathbf{A},\mathbf{B}]_o$ denotes the (ordinary) category whose objects are the **V**-functors from **A** to **B** and whose morphisms are the **V**-natural transformations.

Day and Kelly [6] have provided the result which allows us to make $[\mathbf{A},\mathbf{B}]_o$ into a **V**-category $[\mathbf{A},\mathbf{B}]$. To do so, we need to make use of the concept of an end. For convenience, we repeat the definition here.

3.1 DEFINITION Let **A** and **B** be **V**-categories, and let $T : \mathbf{A}^{op} \otimes \mathbf{A} \to \mathbf{B}$ be a **V**-functor. An *end* of T is a pair (E,π), where $E \in$ Objects(**B**) and π is an Objects(**A**)-indexed family of **V**-natural transformations $\pi_A : E \to T(A,A)$ (regarding $E : \mathbf{A}^{op} \otimes \mathbf{A} \to \mathbf{B}$ as a constant functor) which is final among all such pairs in the sense that if $F \in$ Objects(**B**) and β is an Objects(**A**)-indexed family of **V**-natural transformations $\beta_A : F \to T(A,A)$, then there is a unique **B**-morphism $f : F \to E$ such that the following diagram commutes for each $A \in$ Objects(**A**).



The object E is often denoted by $\int_X T(X,X)$, and $(\int_X T(X,X),\pi)$ is called *the* end of T, since ends are clearly unique up to isomorphism.

3.2 PROPOSITION *Assume that* $\mathbf{V}_o$ *is complete. Then there is a unique (up to isomorphism)* **V** *structure on* $[\mathbf{A},\mathbf{B}]_o$*. More precisely, there is a unique (up to* *isomorphism)* **V**-*category* $[\mathbf{A},\mathbf{B}]$ *and an* Objects($\mathbf{A}$)-*indexed family of* **V**-*functors* $\mathbf{Eval}_A : [\mathbf{A},\mathbf{B}] \to \mathbf{B}$ *such that:*

(a) The underlying category of $[\mathbf{A},\mathbf{B}]$ *is* $[\mathbf{A},\mathbf{B}]_o$.

(b) $\mathbf{Eval}_A(F) = F(A)$, for all $A \in$ Objects($\mathbf{A}$), $F \in [\mathbf{A},\mathbf{B}]_o$.

(c) For $F,G \in$ Objects($[\mathbf{A},\mathbf{B}]$), $[\mathbf{A},\mathbf{B}](F,G) = \int_X \mathbf{B}(F(X),G(X))$, *with the projections of this end given by* $\pi_A(F,G) = \mathbf{Eval}_A(F,G)$.

Proof: See [6, 4.1]. ∎

In **Set**, the end of a functor $T : \mathbf{A} \times \mathbf{A} \to \mathbf{B}$ with $\mathbf{A}$ small is just the coproduct $\coprod_{A \in \mathbf{A}} T(A,A)$, with π_A the A^{th} projection, as is easily verified. Therefore, if $\mathbf{V} = \mathbf{Set}$, $\int_X \mathbf{B}(F(X),G(X)) = \coprod_{A \in \mathbf{A}} \mathbf{B}(F(A),G(A)) = [\mathbf{A},\mathbf{B}]_o(F,G)$, as expected.

For the rest of this section, we shall assume that $\mathbf{V}_o$ is complete, so that the conditions of 3.2 are satisfied.

Our formalization of the concept of lifting a **V**-IFS from **B** to $[\mathbf{A},\mathbf{B}]$ is provided by the following.

3.3 DEFINITION Let $H \subseteq$ Morphisms($\mathbf{B}_o$).

(a) Let $F,G \in [\mathbf{A},\mathbf{B}]$. An $[\mathbf{A},\mathbf{B}]$-morphism $(f : F \to G) = (f : Z \to \int_X \mathbf{B}(F(X),G(X)))$ is a *lifting from* H if each $f_A : Z \to \mathbf{B}(F(A),G(A))$ defined by the end condition

$$\begin{array}{ccc} Z & \xrightarrow{\;f\;} & \int_X \mathbf{B}(F(X),G(X)) \\ & {\scriptstyle f_A} \searrow & \downarrow {\scriptstyle \pi_A} \\ & & \mathbf{B}(F(A),G(A)) \end{array}$$

is in H.

(b) The *lifting of* H *to* $[\mathbf{A},\mathbf{B}]$ is the set of all $[\mathbf{A},\mathbf{B}]_o$ morphisms which are liftings from H, and is denoted $\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(H)$.

(c) If $(\mathbf{E},\mathbf{M})$ is a **V**-IFS for **B**, then $(\mathbf{E},\mathbf{M})$ *lifts to* $[\mathbf{A},\mathbf{B}]$ if $(\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{E}),\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{M}))$ is a **V**-IFS for $[\mathbf{A},\mathbf{B}]$.

We may now state the fundamental result of this section.

3.4 MAIN LIFTING THEOREM FOR RELATIVE FUNCTOR CATEGORIES *Let* $(\mathbf{E},\mathbf{M})$ *be a* **V**-*IFS for* **B**. Then $(\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{E}),\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{M}))$ *is a* **V**-*IFS for* $[\mathbf{A},\mathbf{B}]$.

We prove this theorem with a sequence of three lemmas.

3.5 LEMMA *Let* $K \subseteq \mathrm{Morphisms}(\mathbf{B})$ *be a class of epimorphisms (resp. monomorphsims). Then* $\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(K)$ *is also a class of epimorphisms (resp. monomorphisms).*

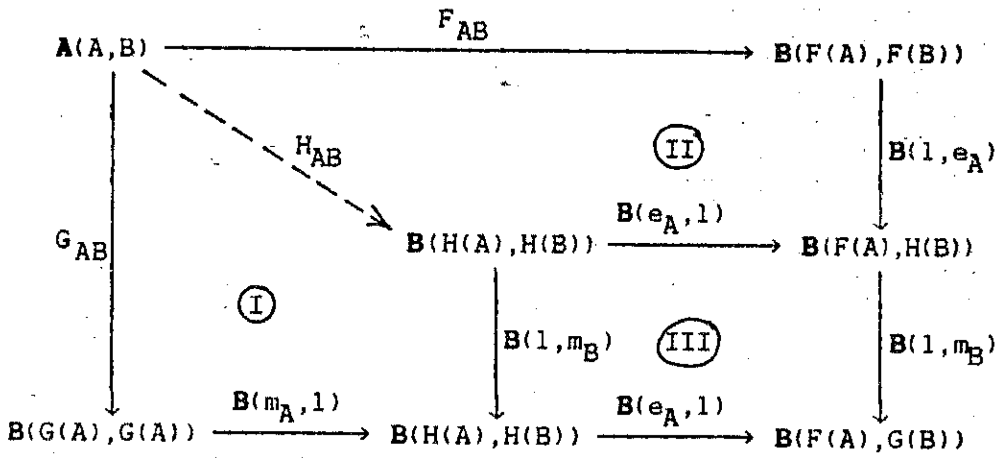
Proof: We will do the monomorphism case, with that for epimorphisms being dual. Let $h : F \to G \in \mathrm{Morphisms}(\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(K))$ and let $H \in \mathrm{Objects}([\mathbf{A},\mathbf{B}])$. Further, let $B \in \mathrm{Objects}(\mathbf{B})$ and let $f_1, f_2 : B \to \int_X \mathbf{B}(H(X),F(X))$ be such that $([\mathbf{A},\mathbf{B}](H,h))\circ f_1 = ([\mathbf{A},\mathbf{B}](H,h))\circ f_2$. For each $A \in \mathrm{Objects}(\mathbf{A})$, consider the diagram below.

$$\begin{array}{ccccc}
B & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & \int_X \mathbf{B}(H(X),F(X)) & \xrightarrow{[\mathbf{A},\mathbf{B}](H,h)} & \int_X \mathbf{B}(H(X),G(X)) \\
 & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_A} \\
 & & \mathbf{B}(H(A),F(A)) & \xrightarrow{\mathbf{B}(H(A),h_A)} & \mathbf{B}(H(A),G(A))
\end{array}$$

where h_A is the projection of h defined by the end condition. The rectangle commutes because of the naturality of the end. Hence $\mathbf{B}(H(A),h_A)\circ\pi_A\circ f_1 = \mathbf{B}(H(A),h_A)\circ\pi_A\circ f_2$, and since $\mathbf{B}(H(A),h_A)$ is a monomorphism by assumption, $\pi_A\circ f_1 = \pi_A\circ f_2$. The universality of the end finally yields $f_1 = f_2$, so that $[\mathbf{A},\mathbf{B}](H,h)$ is a monomorphism. Hence h is a monomorphism. ∎

3.6 LEMMA *Let* $(\mathbf{E},\mathbf{M})$ *be a* **V**-*IFS for* **B**. *Then every* $[\mathbf{A},\mathbf{B}]$ *morphism has a unique* $(\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{E}),\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{M}))$-*factorization.*

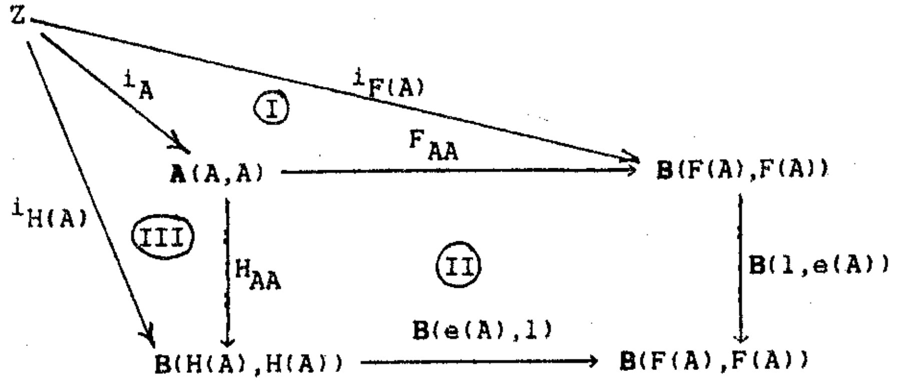
Proof: Let $F,G \in$ Objects($[\mathbf{A},\mathbf{B}]$), and let $\eta : F \to G$ be a **V**-natural transformation. For each $A \in$ Objects(**A**), let (e_A,m_A) be an $(\mathbf{E},\mathbf{M})$-factorization of $\eta(A)$; *i.e.*, $\eta(A) = F(A) \xrightarrow{e_a} H(A) \xrightarrow{m_a} G(A)$. For $A,B \in$ Objects(**A**), consider the following diagram.



(*)

The outer rectangle just asserts the naturality of η, while the inner rectangle III is a pullback in **V**, by R-I5. Hence the dashed-in morphism H_{AB} exists uniquely.

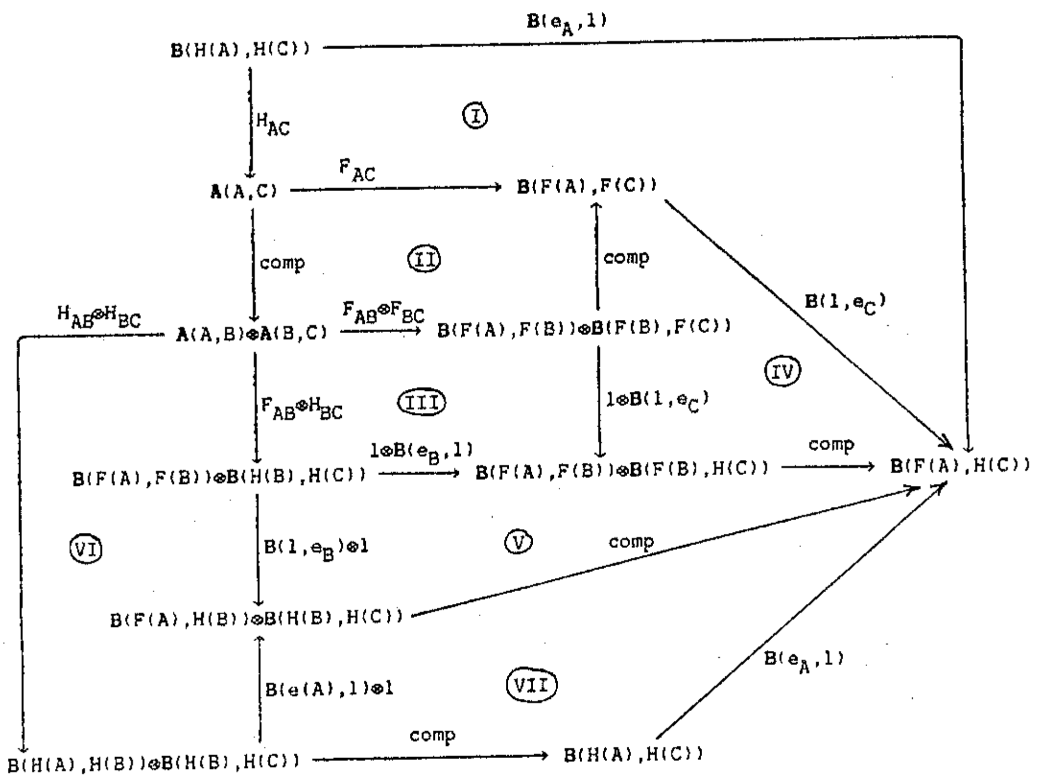
Our task is to show that H_{AB} defines a **V**-functor. Consider the following diagram, for $A \in$ Objects(**A**).



(**)

Triangle I commutes because F is a **V**-functor, and rectangle II is just II of diagram (*) of Lemma 3.6 for $A = B$, hence it also commutes. The lopsided outer rectangle commutes just by definition of identities. Hence $\mathbf{B}(e(A),1)\circ i_{H(A)} = \mathbf{B}(e(A),1)\circ(H_{AA}\circ i_A)$. However, $\mathbf{B}(e(A),1)$ is a monomorphism, so $i_{H(A)} = H_{AA}\circ i_A$; *i.e.*, triangle III commutes. Thus H preserves identities.

We now show that H satifies the functorial composition properties. Let $A, B, C \in$ Objects(**A**), and consider the following enormous diagram in **V**.



The reasons for commutativity of the various pieces are:

I. Rectangle II of diagram (*) of Lemma 3.6.

II. F is a **V**-functor.

III. Left half;: trivial;

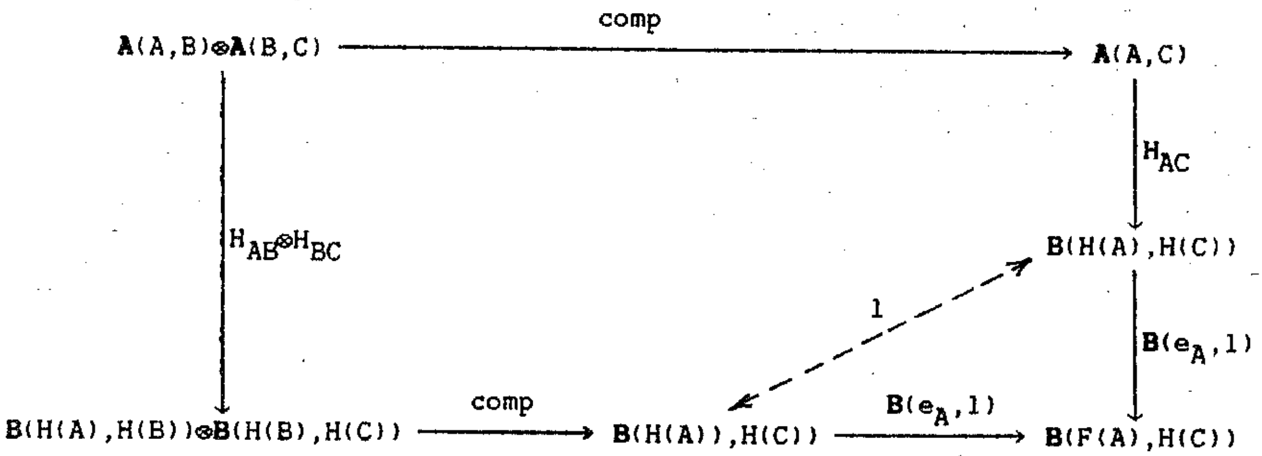
Right half: rectangle II of diagram (*) of Lemma 3.6.

IV, V, and VII. associativity of composition.

VI. Left half: rectangle II of diagram (*) of Lemma 3.6;

Right half: trivial.

Hence the entire diagram commutes. Now we may chase around the outermost path to obtain the following commutative diagram.



However, $\mathbf{B}(e_A,1)$ is a monomorphism, so we may in fact insert the dashed identity and retain commutativity. Hence H satisfies the functorial composition property. Thus H is a **V**-functor.

All we need do to finish the proof is lift the e_A's back to an $e \in [\mathbf{A},\mathbf{B}](F,H)$, and the m_A's back to an $m \in [\mathbf{A},\mathbf{B}](H,G)$. Since both classes are natural by construction, this is trivial, and so we are done. ∎

3.7 LEMMA *Let* e *and* m *be* $[\mathbf{A},\mathbf{B}]$ *morphisms, and suppose that for each* $A \in$ Objects(**A**), the diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\mathbf{B}(e_A,1)} & \cdot \\ {\scriptstyle \mathbf{B}(1,m_A)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{B}(1,m_A)} \\ \cdot & \xrightarrow[\mathbf{B}(e_A,1)]{} & \cdot \end{array}$$

is a pullback in **B**. *Then the diagram*

$$\begin{array}{ccc} \cdot & \xrightarrow{[\mathbf{A},\mathbf{B}](e,1)} & \cdot \\ {\scriptstyle [\mathbf{A},\mathbf{B}](1,m)}\big\downarrow & & \big\downarrow{\scriptstyle [\mathbf{A},\mathbf{B}](1,m)} \\ \cdot & \xrightarrow[{[\mathbf{A},\mathbf{B}](e,1)}]{} & \cdot \end{array}$$

is also a pullback in ***B***.

Proof: The proof is a straightforward exercise in lifting, using the uniqueness of the end to ensure uniqueness of the fill-in. We omit the details. ∎

Proof of Theorem 3.4: Lemmas 3.5 through 3.7 give us properties R-I2, R-I4, and R-I5, respectively. R-I1 and R-I3 are simple consequences of the way that $(\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{E}),\mathrm{Lift}_{[\mathbf{A},\mathbf{B}]}(\mathbf{M}))$ is constructed. ∎

§4. LIFTING TO CATEGORIES OF RELATIVE ALGEBRAS

In this section, we fix a closed category **V** which is assumed to have equalizers. We also fix a **V**-category **A**.

A **V**-*algebraic theory* $\mathbf{T} = [T : \mathbf{A} \to \mathbf{A}, \eta : 1_{\mathbf{A}} \to T, \mu : T^2 \to T]$ in **A** in our terminology is exactly what Bunge [5, 2.1] calls a **V**-standard construction, and what Dubuc [7, p. 60] calls a **V**-monad. Namely, T is a **V**-functor and η and μ are **V**-natural transformations satisfying the usual identities for an algebraic theory. By forgetting the **V**-structure, we get just an ordinary algebraic theory $\mathbf{T}_o = [T_o, \eta_o, \mu_o]$ in $\mathbf{A}_o$. An *algebra* $[A,\xi]$ of **T** is exactly an algebra of $\mathbf{T}_o$ in the usual sense. $\mathbf{A}_o^{\mathbf{T}_o} = (\mathbf{A}^{\mathbf{T}})_o$ denotes the category of **T** algebras and algebra homomorphisms.

Throughout this section, fix a V-algebraic theory $\mathbf{T} = [T,\eta,\mu]$.

The category $(\mathbf{A}^{\mathbf{T}})_o$ has a natural V-structure, hence justifying the notation. This is embodied in the following proposition.

4.1 PROPOSITION $(\mathbf{A}^{\mathbf{T}})_o$ *can be naturally elevated to a* **V**-*category* $\mathbf{A}^{\mathbf{T}}$ *by defining for* $[A,\xi], [B,\theta] \in \text{Objects}((\mathbf{A}^{\mathbf{T}})_o)$, $U^{\mathbf{T}}_{[A,\xi][B,\theta]} : \mathbf{A}^{\mathbf{T}}([A,\xi],[B,\theta]) \to \mathbf{A}(A,B)$ *to be the equalizer in* **V** *of the following two morphisms.*

$$\mathbf{A}(A,B) \xrightarrow{T_{AB}} \mathbf{A}(T(A),T(B)) \xrightarrow{\mathbf{A}(T(A),\theta)} \mathbf{A}(T(A),B)$$
$$\mathbf{A}(A,B) \xrightarrow{\mathbf{A}(\xi,B)} \mathbf{A}(T(A),B)$$

$U^{\mathbf{T}}$ *is, of course, the* **V**-*version of the forgetful functor* $[A,\xi] \mapsto A$.

Proof: See [5, Prop. 2.2]. ∎

4.2 DEFINITION Let $H \subseteq \text{Morphisms}(\mathbf{A})$. The *lifting* of H to $\mathbf{A}^{\mathbf{T}}$ is $\{ f \in \text{Morphisms}(\mathbf{A}^{\mathbf{T}}) \mid U^{\mathbf{T}}(f) \in H \}$, and is denoted $\text{Lift}_{\mathbf{T}}(H)$.

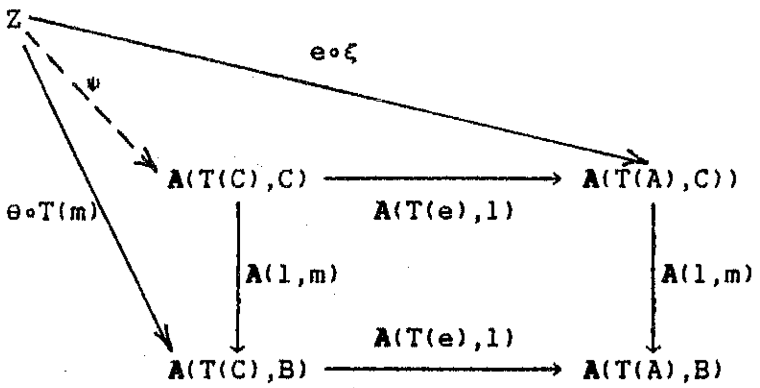
Our fundamental lifting result for algebras is the following.

4.3 MAIN LIFTING THEOREM FOR CATEGORIES OF RELATIVE ALGEBRAS *Let* (**E**,**M**) *be a* **V**-*IFS for* **A** *with the property that* T *preserves* **E**. *Then* ($Lift_T$(**E**),$Lift_T$(**M**)) *is a* **V**-*IFS for* $\mathbf{A}^\mathbf{T}$.

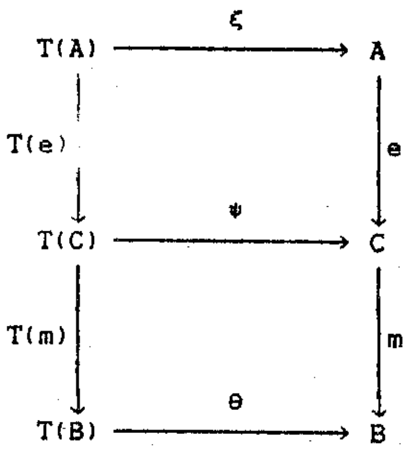
Note that this is a true generalization of the result for ordinary algebraic thories [16, Ch. 3, Prop. 4.17]. The following two propositions provide the core of the proof of the lifting theorem.

4.4 PROPOSITION *Let* (**E**,**M**) *be a* **V**-*IFS for* **A**, *and suppose further that* T *preserves* **E**. *Then* ($Lift_T$(**E**),$Lift_T$(**M**)) *satisfies* R-I4.

Proof: Let $h : [A,\xi] \to [B,\theta]$ be a morphism in $\mathbf{A}^\mathbf{T}$, and let $A \xrightarrow{e} C \xrightarrow{m} B$ be an (**E**,**M**) factorization of the underlying (via $U^\mathbf{T}$) **A**-morphism. Consider the following diagram in **V**.



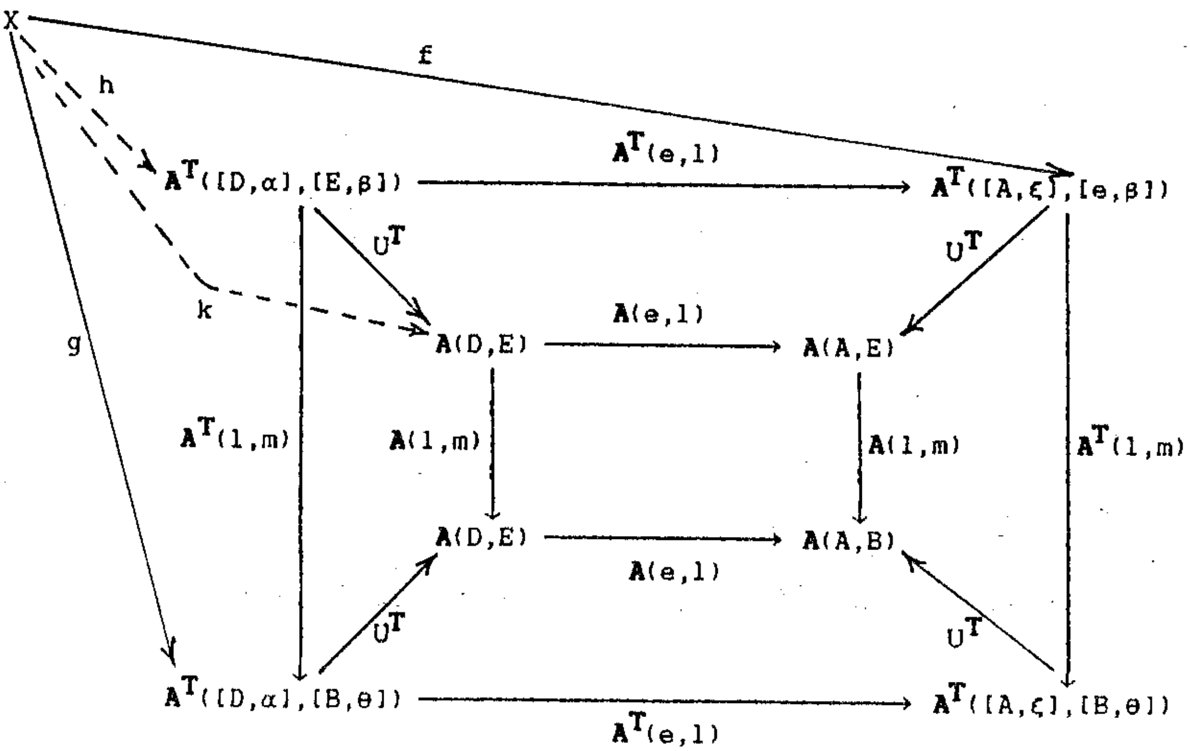
By R-I5 for (**E**,**M**) and the hypothesis that T preserves **E**, the rectangle is a pullback. Hence the dashed morphism ψ exists uniquely, and so we have the commutativity of the following diagram in $\mathbf{A}_o$.



The verification that [C,ψ] is a T-algebra is no different than in the ordinary case. See [16, Ch. 3, Prop. 4.17]. ∎

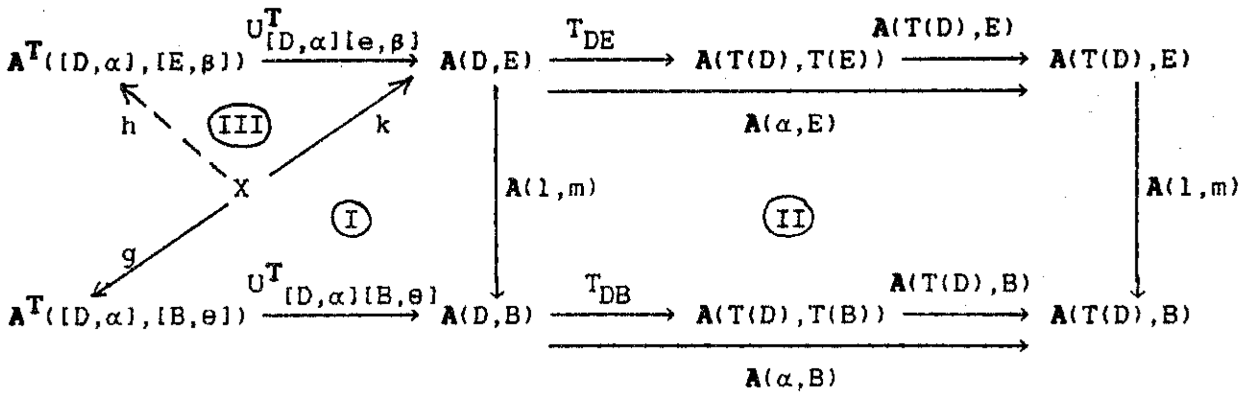
4.5 PROPOSITION *Let* (**E**,**M**) *be a* **V**-*IFS for* **A**, *and suppose that* T *preserves* **E**. *Then* (Lift_T(**E**),Lift_T(**M**)) *satisfies* R-I5.

Proof: Let [A,ξ], [B,θ], [D,α], and [E,β] be T-algebras with e : [A,ξ] → [D,α] ∈ Lift_T(**E**) and m : [E,β] → [B,θ] ∈ Lift_T(**M**). Consider the following diagram in **V**.



(Note: For simplicity, the (obvious) parameters of the $U^{\mathbf{T}}$'s have been omitted.) We must show that the outer rectangle is a pullback; that is, given f and g, we must show that h exists uniquely.

First of all, note that because (**E**,**M**) is a **V**-IFS, the inner rectangle is a pullback, so k exists uniquely. Furthermore, the four trapezoids connecting the inner and outer rectangles commute because $U^{\mathbf{T}}$ is a functor. Thus, we may construct the following diagram.



I commutes because k is the unique fill-in of the inner pullback of the previous diagram. Since $U^{\mathbf{T}}_{[D,\alpha][E,\beta]}$ is the equalizer of the lower parallel arrows, we must have that $(\mathbf{A}(T(D),B)\circ T_{DB}\circ\mathbf{A}(1,m))\circ k = (\mathbf{A}(\alpha,B)\circ\mathbf{A}(1,m))\circ k$. Now the rectangles II commute upon selecting either the upper arrows of the parallel arrows (because **m** is a **T**-algebra morphism) or the lower arrows of the parallel pairs (because composition of morphisms is associative). Hence $\mathbf{A}(1,m)\circ\mathbf{A}(T(D),E)\circ T_{DE}\circ k = \mathbf{A}(1,m)\circ\mathbf{A}(\alpha,E)\circ k$, and since $\mathbf{A}(1,m)$ is a monomorphism, $\mathbf{A}(T(D),E)\circ T_{DE}\circ k = \mathbf{A}(\alpha,E)\circ k$. Thus, since $U^{\mathbf{T}}_{[D,\alpha][E,\beta]}$ is the equalizer of $\mathbf{A}(T(D),E)\circ T_{DE}$ and $\mathbf{A}(\alpha,E)$, we have the existence of the unique fill-in h making III commute, as required. ∎

Proof of Theorem 4.3: Propositions 4.4 and 4.5 give use conditions R-I4 and R-I5, respectively. R-I1, R-I2, and R-I3 are trivially satisfied by $(\mathrm{Lift}_T(\mathbf{E}),\mathrm{Lift}_T(\mathbf{M}))$. ∎

§5. LIFTING IFS's TO V-IFS's

Since many examples of ordinary IFS's are known, it is useful to know exactly when these are also **V**-IFS's for a suitable closed category **V**. In this section, we develop sufficient conditions for lifting of an ordinary IFS to the **V** case. We also present several examples. Throughout this section, **V** is assumed to be a closed category, with any further properties explicitly indicated.

First, we recall the special properties of a **V**-IFS. Let **A** be a **V**-category, and let (**E**,**M**) be an IFS for $\mathbf{A}_0$. There two ways in which (**E**,**M**) can fail to be a **V**-IFS for **A**.

(i) R-I2 can be violated. That is, **E** can contain epimorphisms which are not **V**-epimorphisms or **M** can contain monomorphisms which are not **V**-monomorphisms.

(ii) R-I5 can be violated. That is, for some $(e,m) \in \mathbf{E} \times \mathbf{M}$,

$$\begin{array}{ccc} \cdot & \xrightarrow{\mathbf{A}(e,1)} & \cdot \\ {\scriptstyle \mathbf{A}(1,m)}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(1,m)} \\ \cdot & \xrightarrow[\mathbf{A}(e,1)]{} & \cdot \end{array}$$

can fail to be a pullback in **V** (even though it must be a pullback in **Set**).

Recall that in any category **V**, an object S is a *separator* if the functor $\mathbf{V}(S,-) : \mathbf{V} \to \mathbf{Set}$ is faithful. If **V** has copowers, this is equivalent to the condition that for every object X of **V**, there is a copower ^{I}S of S and an epimorphism $e : {}^{I}S \to X$ [13, 19.6]. This leads to the more restrictive notion of retract-separator. In a category **V** with copowers, an object R is a *retract-separator* for **V** if for every object X of **V**, there is a copower ^{I}R of R and a retract $r : {}^{I}R \rightarrow X$ [13, 19.7]. Each of these properties is useful in characterizing those IFS's which are also **V**-IFS's, as is illustrated by the following two results.

5.1 PROPOSITION *Let* **A** *be a* **V**-*category, and let* (**E**,**M**) *be an IFS for* **A**. *If* Z *is a separator for* **V**, *then* (**E**,**M**) *satisfies* R-I2.

Proof: Faithful functors reflect monomorphisms [13, 12.8]. ∎

5.2 THEOREM *Let* **A** *be a* **V**-*category with copowers, and let* (**E**,**M**) *be an IFS for* **A**. *If* Z *is a retract-separator for* **V**, *then* (**E**,**M**) *is a* **V**-*IFS.*

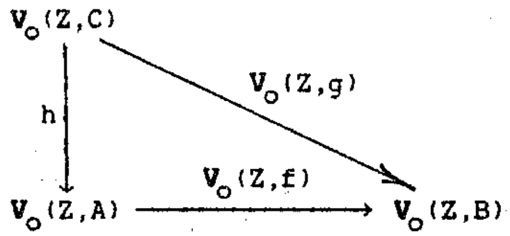
Proof: First of all, every retract separator is a separator, so (**E**,**M**) satisfies R-I2. To show that R-I5 holds, it suffices to demonstrate that **V**(Z,-) : **V** → **Set** reflects pullbacks, since I5" holds. However, **V**(Z,-) reflects all limits in the case that **V**(Z,-) is a retract separator [13, 29.4], so we are done. ∎

5.3 EXAMPLE *Let* R *be any commutative ring with identity, and let* R-mod *denote the category of* R-*modules.* R-mod *is a closed monoidal category with the usual tensor product. Furthermore, for every* R-mod *category* **A**, *every IFS is an* (R-mod)-*IFS.*

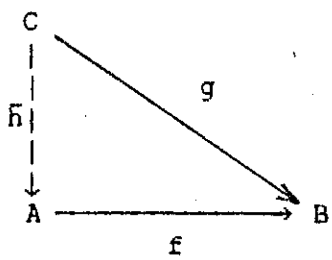
Proof: The verification that R-mod forms a closed monoidal category is standard. Furthermore, every R-module is the homomorphic image (= retract image) of a free R-module (= copower of R) [9, Ch. 3, Thm 1.2, Cor]. Hence the result follows from Theorem 5.2. ∎

While is very often the case that Z is a separator for **V**, the further condition that it be a retract separator is much stronger and often not met in practice. We therefore provide the following additional result for ensuring that R-I5 is satisfied.

5.4 DEFINITION A morphism $f : A \to B$ in **V** is called an *embedding* [13, 34G] provided that for each **V**-object C, **V**-morphism $g : C \to B$ and each **Set** morphism $h : \mathbf{V}_o(Z,C) \to \mathbf{V}_o(Z,A)$ such that



commutes, there is a unique **V**-morphism $\bar{h} : C \to A$ such that the following diagram commutes.



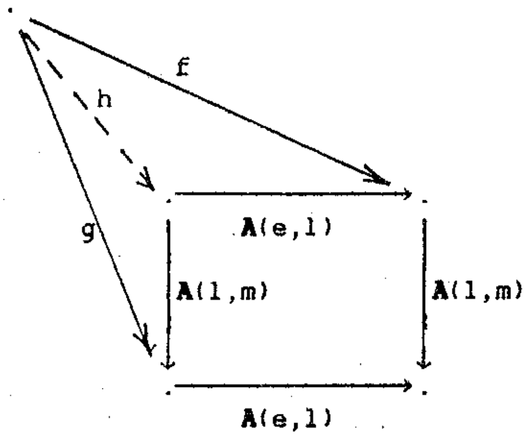
5.5 LEMMA *Let* **A** *be a* **V**-*category, and let* (**E**,**M**) *be an IFS for* **A**. *If either*

(i) each **A**$(e,1)$ *is an embedding for each* $e \in$ **E**, *or*

(ii) each **A**$(1,m)$ *is an embedding for each* $m \in$ **M**,

then (**E**,**M**) *satisfies* R-I5.

Proof: We first construct the pullback diagram in **Set**, where we know the fill-in h shown below to exist.



Now, by the very definition of embedding, we can lift h up to **V** using the fact that either **A**(e,1) or **A**(1,m) is an embedding in **V**. ∎

5.6 THEOREM *Let* **A** *be a* **V**-*category, and let* (**E**,**M**) *be an IFS for* **V**. *Suppose further that* Z *is a separator for* **V**. *Then if either* **A**(e,1) *is an embedding for each* e ∈ **E** *or* **A**(1,m) *is an embedding for each* m ∈ **M**, (**E**,**M**) *is a* **V**-*IFS*. ∎

Theorem 5.6 turns out to be most useful in practice. In the next section, we show how it may be applied to examples in the category of Banach spaces. See also [12] for applications of this theorem to problems in (C_o) semigroup factorization and realization theory.

§6. EXAMPLES IN THE CATEGORY OF BANACH SPACES

The category of Banach spaces provides a convenient showcase for examples of our results. On the one hand, its basic characteristics are widely known, and additional requisites are easy to develop. On the other hand, easily understandable examples of IFS's which are also relative IFS's, as well as IFS's which are not, also exist.

Fix **K** to be either the field **R** of real numbers or the field **C** of complex numbers. $\mathbf{Ban}_o$ denotes the (ordinary) category whose objects are the Banach spaces over **K** and whose morphisms are those linear mappings whose norm does not exceed one. (We use the usual supremum norm: $\|f\| = \sup\{ \|f(x)\| \mid \|x\| \leq 1 \}$.)

To impose a closed category structure, we proceed as follows. Let **Ban**(E,F) denote the Banach space of *all* bounded linear mappings from E to F, equipped with the supremum norm defined above. $E \hat{\otimes} F$ denotes the complete normed tensor product of E and F [17, Ch. 43]. Regard **K** as a Banach space with the usual absolute value norm. We then have the following.

6.1 PROPOSITION $\mathbf{Ban}_o$ *becomes a closed monoidal category* **Ban** *when equipped with tensor product* $-\hat{\otimes}-$, *unit element* **K**, *and the hom structure* **Ban**(E,F) *as denoted above.*∎

We omit the rather straightforward proof of the above result. A somewhat more detailed discussion of the closed monoidal properties of this category can be found in [18].

In the following examples, we shall be developing IFS's and relative IFS's for **Ban**, regarding it as a closed category over itself. (In terms of our earlier notational conventions of sections 2 and 5, we now fix **V** = **A** = **Ban**.)

To understand the following examples, it is necessary to clarify some notation. A *true quotient* in **Ban** is a mapping $f : E \to F$ which is surjective and for which F carries the quotient norm (*i.e.*, for any $y \in F$, $\|y\| = \inf\{ \|x\| \mid y = f(x) \}$). A *scaled quotient* is of the form αf, where f is a true quotient and $0 < \alpha \leq 1$. Similarly, a *true isometric embedding* in **Ban** is a mapping $f : E \to F$ which is an isometry onto a closed subspace of F, while a *scaled isometric embedding* is of the form αf, where f is a true isometric embedding and $0 < \alpha \leq 1$.

The reader is cautioned that not every surjection is a scaled quotient, even in the light of the open mapping theorem [17, Thm. 17.1]. For example, let $f : \mathbf{K}^2 \to \mathbf{K}^2$ (with $\mathbf{K}^2$ carrying the usual ℓ^2 norm) be given by $(x,y) \mapsto (x,y/2)$. Then f is surjective and, in fact, $\|f\| = 1$, yet f is neither a true quotient nor a scaled quotient. At the same time, f is also an example of a topological embedding of norm 1 which is not an isometric embedding, either true or scaled.

Finally, we introduce one more item of notation. We will need to deal with the class of all **Ban**-morphisms $f : E \to F$ with a dense image (which we call *dense mappings*) and with the class of all **Ban**-morphisms which are injective (which we call simply *injections*). However, we shall also need to deal with the subclass of the dense mappings consisting precisely of those whose norm is either 1 or 0. We shall call these the $\|0/1\|$*-dense mappings*. Similarly, we need the subclass of injections consisting of those whose norm is either 1 or 0; we call these the $\|0/1\|$*-injections*.

6.2 PROPOSITION Each of the following is an ordinary IFS for **Ban**.

(a) (true quotients, injections)

(b) (dense mappings, true isometric embeddings)

(c) (scaled quotients, $\|0/1\|$*-injections)*

(d) (‖0/1‖-dense mappings, scaled isometric embeddings).

Proof: (a) and (b) are well known. Given $f : E \to F$, for (a) we factor f as $E \xrightarrow{e} E/\ker(f) \xrightarrow{m} F$ with $E/\ker(f)$ carrying the quotient norm. For (b), we factor f as $E \xrightarrow{e} \overline{f(E)} \xrightarrow{m} F$, with $\overline{f(E)}$ denoting the closure of F(E) in F, equipped with the norm inherited from F. (c) and (d) are just scaled versions of (a) and (b) respectively, in which we scale the norms so that the component in the factorization which has norm 1 is changed. Given $f : E \to F$ and a (true quotients, injections) factorization $E \xrightarrow{e} G \xrightarrow{m} F$, we get the (scaled quotients, ‖0/1‖-injections) factorization, in the case that $\|m\| \neq 0$, by replacing e by $\|m\| \cdot e$ and replacing m by $m/\|m\|$, with G remaining the same. If $\|m\| = 0$, the (scaled quotients, ‖0/1‖-injections) factorization coincides with the (true quotients, injections) factorization, as in this case $\|e\| = 0$ also. The (‖0/1‖-dense mappings, scaled isometric embeddings) factorization is obtained in an entirely analogous way from the (dense mappings, true isometric embeddings) factorization. In each case, verification of the IFS properties is trivial. ∎

We shall now prove that while (a) and (b) of the above proposition are **Ban**-IFS's, (c) and (d) are not.

6.3 PROPOSITION *Every IFS for* **Ban** *satisfies* R-I2.

Proof: This follows directly form Proposition 5.1 and the fact that **K** is a separator for **Ban**. ∎

It will thus be shown that R-I5 is crucial in determining whether or not a given IFS is a **Ban**-IFS. To show that the IFS's (a) and (b) of Proposition 6.2 satisfy R-I5, we must use Lemma 5.5.

6.4 LEMMA *A* **Ban**-*morphism is an embedding in the sense of Definition 5.4 if* *and only if it is a true isometric embedding.*

Proof: First note that since for any Banach space F, **Ban**(K,F) ≅ F, we may restate Definition 5.4 as follows. The **Ban**-morphism $f : A \to B$ is an embedding if for any Banach space C, **Ban**-morphism $g : C \to B$, and function $h : C \to A$ such that $f \circ h = g$ as functions, h must in fact be a **Ban**-morphism.

Thus, let $f : A \to B$ be a true isometric embedding, with $g : C \to B$ a **Ban**-morphismand $h : C \to A$ a function with $f \circ h = g$. Since f is a true isometric embedding, we must have that $\|h(x)\| = \|g(x)\|$ for all $x \in C$. Since h is clearly linear (since f is injective), it follows that h is a **Ban**-morphism with $\|h\| = \|g\|$.

Conversely, suppose that $f : A \to B$ is any **Ban**-morphism. Let C be the closure of f(A) in B, with $g : C \to B$ the corresponding true isometric embedding. There is a unique function $h : C \to A$ such that $f \circ h = g$ only if f is injective. In that case, h is given by $x \mapsto f^{-1}(g(x))$. This h is always linear, but $\|h(x)\| = \|f^{-1}(g(x))\| = \|f^{-1}(x)\|$, since g is a true isometric embedding. Thus, for h to be a **Ban**-morphism we must have $\|f^{-1}(x)\| \leq \|x\|$; *i.e.*, f must be a true isometric embedding. ∎

6.5 LEMMA *(a) Let* $e : E \to F$ *be a true quotient in* **Ban**. *Then for any Banach space* G, **Ban**(e,1) : **Ban**(F,G) → **Ban**(E,G) *is an embedding in the sense of Definiton 5.4.*

(b) Let $m : E \to F$ *be a true isometric embedding in* **Ban**. *Then for any Banach space* G, **Ban**(1,m) : **Ban**(G,E) → **Ban**(G,F) *is an embedding in the sense of Definition 5.4.*

Proof: We show only (a); (b) is similar. In view of Lemma 6.4, it is equivalent to show that **Ban**(e,1) is a true isometric embedding. Now, for any $f \in$ **Ban**(F,G), $\|f \circ e\| = \sup_{x \in E}\{ \|f \circ e(x)\| \mid \|x\| \leq 1 \} = \sup_{y \in F}\{ \|f(y)\| \mid \|y\| \leq 1 \}$,

since e is a true quotient. Hence $\|f \circ e\| = \|f\|$. (Note that $\|e\| = 0$ if and only if $F = 0$, in which case $\|f\| = 0$, so this causes no problem). ∎

6.6 THEOREM *The following are* **Ban**-*IFS's.*

(a) (true quotients, injections)

(b) (dense mappings, true isometric embeddings).

Proof: Using Lemmas 5.5 and 6.5, we have that both IFS's satisfy R-I5. Since by Proposition 6.3 they satisfy R-I2, we have that they are **Ban**-IFS's. ∎

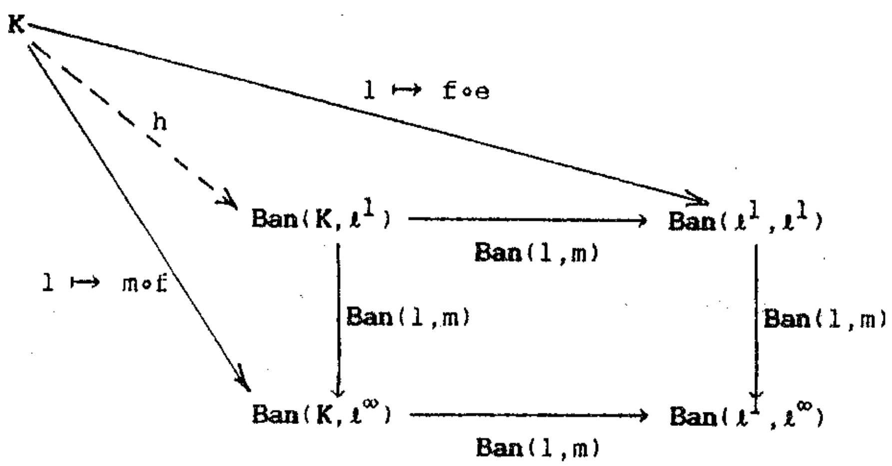
We now turn to the problem of showing tha the IFS's of (c) and (d) of Proposition 6.2 are not **Ban**-IFS's. To do so, we explicitly demonstrate the failure of the pullback condition R-I5.

6.7 THEOREM *The following IFS's for* **Ban** *are not* **Ban**-*IFS's.*

(a) (scaled quotients, $\|0/1\|$*-injections)*

(b) ($\|0/1\|$*-dense mappings, scaled isometric embeddings).*

Proof: We show (a); (b) is similar. We let ℓ^1 and ℓ^∞ have their usual meanings [17, p. 101]. Define $e : \ell^1 \to \mathbf{K}$ by $(x_0, x_1, x_2, .. x_k, ...) \mapsto x_0/2$. e is easily seen to a scaled quotient with $\|e\| = 1/2$. Define $m : \ell^1 \to \ell^\infty$ to be the natural injection $(x_0, x_1, x_2, .., x_k, ...) \mapsto (x_0, x_1, x_2, .., x_k, ...)$. Finally, define $f : \mathbf{K} \to \ell^1$ by $x \mapsto (x, x, 0, 0, 0, .. 0, ...)$, and note that $\|f\|$ $= 2$. Now consider the diagram below.



Note that $\|m \circ f\| = 1$ and $\|f \circ e\| = 1$, so the two outer arrows are of norm 1 and so are legal **Ban**-morphism. However, the unique fill-in h (in **Set**) is $1 \mapsto f$, which has norm 2 since $\|f\| = 2$. Since this is the only possible fill in in **Ban**, we have that no legal h (in **Ban**) can exist. Hence R-I5 is not satisfied, and so (scaled quotients, $\|0/1\|$-injections) is not a **Ban**-IFS, even though it is an ordinary IFS, by Proposition 6.2. ∎

REFERENCES

1. M. A. Arbib and E. G. Manes, Foundations of system theory: decomposable systems, *Automatica - J. IFAC*, **10**(1974), pp. 285-302.

2. M. A. Arbib and E. G. Manes, Adjoint machines, state-behavior machines, and duality, *J. Pure Appl. Algebra*, **6**(1975), pp. 313-344.

3. M. A. Arbib and E. G. Manes, *Arrows, Structures, and Functors: The Categorical Imperative*, Academic Press, New York, 1975.

4. E. S. Bainbridge, *A Unified Minimal Realization Theory with Duality*, Dissertation, Univerisity of Michigan, 1972.

5. M. C. Bunge, Relative functor categories and categories of algebras, *J. Algebra*, **11**(1969), pp. 64-101.

6. B. Day and G. M. Kelly, Enriched functor categories, *Reports of the Midwest Category Seminar III*, S. MacLane, editor, Springer-Verlag, New York, 1969, pp. 178-191.

7. E. J. Dubuc, *Kan Extensions in Enriched Category Theory*, Springer-Verlag, New York, 1970.

8. S. Eilenberg and G. M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra*, edited by S. MacLane *et al*, Springer-Verlag, New York, 1966.

9. J. K. Goldhaber and G. Ehrlich, *Algebra*, MacMillan, New York, 1970.

10. S. J. Hegner, Linear decomposable systems in continuous time, *SIAM J. Math. Anal.*, **12**(1981), pp. 243-273.

11. S. J. Hegner, Algebraic characterization of classes of (C_o) semigroups with applications to system theory, submitted for publication.

12. S. J. Hegner, Algebraic theories of (C_o) semigroups, to appear.

13. H. Herrlich and G. E. Strecker, *Category Theory*, Allyn and Bacon, Boston, 1973.

14. G. M. Kelly, Adjunction for enriched categories, *Reports of the Midwest Category Seminar III*, S. Maclane, editor, Springer-Verlag, New York, 1969, pp. 166-177.

15. G. M. Kelly, *Basic Concepts of Enriched Category Theory*, Cambridge University Press, New York, 1982.

16. E. G. Manes, *Algebraic Theories*, Springer-Verlag, New York, 1976.

17. F. Treves, *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, New York, 1967.

18. J. Wick-Negrepontis, Duality of functors in the category of Banach spaces, *J. Pure Appl. Algebra*, 3(1973), pp. 119-131.