# An Algebraic Approach to Continuous-Time Linear Systems Defined over Banach Spaces

by

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Abstract

The class of continuous-time linear systems whose dynamics are described by $(C_o)$ semigroups do not admit a definition of natural behavior in terms of universal constructions. To capture the concept of behavior for these systems, they are embedded in a larger class of systems, called $\mathcal{M}$-systems. The basic categorical aspects of behavior and canonical realization is presented for $\mathcal{M}$-systems.

## 1. Introduction

This article is an extended summary without proofs. Full details of all results presented herein will appear in [9].

A problem which has long been studied in system theory is that of realization, which provides a local (or internal) description of a system, given its behavior over time. For many classes of systems, the definition of behavior is simple and intuitive, so the process of recovering a behavior from an internal description does not seem particularly interesting. Nonetheless, it is known that the usual definition of behavior for many classes of systems arises naturally from universal constructions [1,2]. In the theory of continuous-time linear systems, the definition of behavior is not so obvious, and so a variety of definitions have appeared in the literature. While universal constructions have been used to define the behavior of more general classes of continuous-time linear systems [7,8], none are entirely within the framework of Banach spaces. Since the bulk of research in infinite-dimensional continuous-time linear systems is done within the domain of Banach spaces, it seems natural to attempt to define the behaviors of such systems using universal constructions. An investigation of such definitions is pursued in the first part of this paper.

Of course, the actual process of realization must also be a part of any theory, and a realization theory for the systems developed in this report is also presented. Space limitations do not permit the presentation of a duality theory, which must be postponed to the full report [9].

In this summary, the use of category theory has been kept to only that absolutely necessary to present the concepts. For definitions not presented here, the reader is referred to [3].

## 2. The Behavior Problem for Smooth Linear Systems

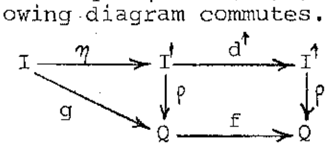
Throughout this paper, $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) denotes the nonnegative (resp. nonpositive) reals. $\mathbb{K}$ is fixed to be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. BAN denotes the category whose objects are the Banach spaces over $\mathbb{K}$, and whose morphisms are the linear maps which do not increase the norm. $(C_o)$ semigroups are defined in the sense of contraction semigroups of class $(C_o)$ in the sense of Yosida [13]; i.e., if $T$ is a $(C_o)$ semigroup, then $\|T(t)\| \leq 1$ for all $t \in \mathbb{R}_+$. $\mathfrak{g}_T$ denotes the infinitesimal generator of $T$. Since the infinitesimal generator uniquely determines the semigroup, the notations $(E,T)$ and $(E,\mathfrak{g}_T)$ will be used interchangeably to denote the semigroup $T$. $(C_o)$-DYN denotes the category whose objects are the $(C_o)$ semigroups; a morphism $k: (E,T) \to (F,S)$ is a BAN morphism $k: E \to F$ such that $k \circ T(t) = S(t) \circ k$ for all $t \in \mathbb{R}_+$. This is equivalent to requiring that $k \circ \mathfrak{g}_T = \mathfrak{g}_S \circ k$ whenever both sides are defined.

Smooth linear systems are those which have $(C_o)$ semigroups for their dynamics. More specifically, a smooth linear system is a 6-tuple $M = (Q,f,I,g,Y,h)$, where $Q$, $I$, and $Y$ are Banach spaces (the state space, input space, and output space, respectively), $g \in \text{BAN}(I,Q)$ (the input map), $h \in \text{BAN}(Q,Y)$ (the output map), and $f$ (the state-transition map) is the infinitesimal generator of a $(C_o)$ semigroup on $Q$. The dynamics of $M$ are thought of as governed by the equations

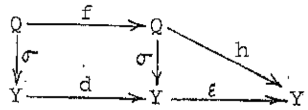
$$\frac{dq(t)}{dt} = f(q(t)) + g(i(t))$$

$$y(t) = h(q(t)).$$

Given a Banach space $I$, a free $(C_o)$ dynamics over $I$ is a $(C_o)$ semigroup $(I^{\dagger}, d^{\dagger})$ and an $\eta \in \text{BAN}(I, I^{\dagger})$ such that for any $(C_o)$ semigroup $(Q,f)$ and $g \in \text{BAN}(I,Q)$, there is a unique $\rho \in \text{BAN}(I^{\dagger},Q)$ such that the following diagram commutes.




Dually, given a Banach space $Y$, a cofree $(C_o)$ dynamics over $Y$ is a $(C_o)$ semigroup $(Y_{\uparrow}, d_{\uparrow})$ and an $\varepsilon \in \text{BAN}(Y_{\uparrow}, Y)$ such that for any $(C_o)$ semigroup $(Q,f)$ and $h \in \text{BAN}(Q,Y)$, there is a unique $\sigma \in \text{BAN}(Q,Y_{\uparrow})$ such that the following diagram commutes.




Now given a smooth linear system $M = (Q,f,I,g,Y,h)$, if both of the above constructions exist, the behavior of M is defined to be $\sigma \circ \rho$, with $\rho$ the reachability map and $\sigma$ the observability map. This approach, when applied in the discrete-time case (replacing f with a map representing one-step transition) (see [1]), yields the $k[z]$-module approach of Kalman [10]. It has also been applied to certain classes of continuous-time linear systems with success [7,8]. However, it fails within the current framework.

THEOREM For $I \neq 0$ (the trivial Banach space), free $(C_0)$ dynamics over I do not exist.

Thus, another definition for behavior of smooth linear systems must be found. The approach taken here is similar to that of Bensoussan and Kamen [4] in that the class of smooth linear systems is embedded into a larger class of systems, and behaviors and realizations are considered for this larger class. However, the approach taken here uses universal constructions, so that the definition of behavior is based upon the definition of internal description.

## 3. <u>$\mathcal{M}$-Systems</u>

Let $\mathcal{C}_0$ denote the set of all continuous functions $f: \mathbb{R}_+ \to \mathbb{K}$ which are bounded and which vanish at infinity. $\mathcal{C}_0$ is a Banach space with the norm $f \mapsto \sup\{f(x) \mid x \in \mathbb{R}_+\}$. Let $\mathcal{M} = \mathcal{C}_0'$, the strong (Banach) dual of $\mathcal{C}_0$. The following characterization of $\mathcal{M}$ may be found in [5].

THEOREM $\mathcal{M}$ is precisely the set of all bounded measures on $\mathbb{R}$ with support contained in $\mathbb{R}_+$. Under convolution $*$, $\mathcal{M}$ is a commutative Banach algebra with unit $\delta_0$, the Dirac measure at 0.

An <u>$\mathcal{M}$-Banach module</u> is a pair $(E,b)$, where E is a Banach space and $b: \mathcal{M} \times E \to E$ is a bilinear map such that E is a module over $\mathcal{M}$ with $\|b(\mu,e)\| \le \|\mu\| \cdot \|e\|$ for all $\mu \in \mathcal{M}$, $e \in E$. $\mathcal{M}$-MOD denotes the category whose objects are the $\mathcal{M}$-Banach modules and whose morphisms are module homomorphisms $k: (E,b) \to (F,c)$ such that $k \in BAN(E,F)$.

The class of $\mathcal{M}$-Banach modules will be used as the dynamics for a class of linear systems which includes the smooth linear systems. Therefore, it is necessary to show that each smooth linear system has a natural representation as an $\mathcal{M}$-Banach module. Let $\delta_t \in \mathcal{M}$ denote the Dirac measure at $t \in \mathbb{R}_+$. Let $\Delta$ denote the subspace of $\mathcal{M}$ spanned by these Dirac measures. Given a $(C_0)$ semigroup $(E,T)$, define $\hat{T}: \Delta \times E \to E$ by $(\delta_t, e) \mapsto T(t)e$, extended by linearity. Defined as such, it is easy to see that $\hat{T}$ is bilinear, and that the following is also true.

THEOREM Let $(E,T)$ be a $(C_0)$ semigroup. Then T has an extension $\tilde{T}: \mathcal{M} \times E \to E$ which turns E into an $\mathcal{M}$-Banach module.

Unfortunately, $\Delta$ is not dense in $\mathcal{M}$ for the norm topology, so there are in general many such extensions. However, there is one extension which seems most natural in the sense that it is also separately continuous for a weaker but nonetheless natural topology on $\mathcal{M}$. Let $\mathcal{B}$ denote the Banach space of all bounded uniformly continuous functions $\mathbb{R}_+ \to \mathbb{K}$ with the sup norm. It is not difficult to verify that there are natural isometric embeddings $\mathcal{C}_0 \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{M}'$. Hence there is a natural separating dual pairing $\langle \mathcal{M}, \mathcal{B} \rangle$. Let $\mathcal{T}_c$ denote the locally convex topology on $\mathcal{M}$ whose neighborhood base at 0 is given by polars of compact subsets of $\mathcal{B}$.

THEOREM (a) $\Delta$ is dense in $\mathcal{M}$ for the topology $\mathcal{T}_c$. (b) If $(E,T)$ is a $(C_0)$ semigroup, the natural map $\hat{T}: \Delta \times E \to E$ is separately continuous for the topology $\mathcal{T}_c$ on E, and so extends to a unique $\tilde{T}: \mathcal{M} \times E \to E$ which is also an $\mathcal{M}$-Banach module action (for the norm topology on $\mathcal{M}$).

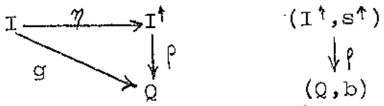
$\tilde{T}$ will always denote this particular extension. $(E,\tilde{T})$ is called the $\mathcal{M}$-Banach module associated with $(E,T)$.

An <u>$\mathcal{M}$-system</u> is a 6-tuple $M = (Q,f,I,g,Y,h)$, where Q, I, and Y are Banach spaces (the state space, input space, and output space, respectively), $g \in BAN(I,Q)$ (the input map), $h \in BAN(Q,Y)$ (the output map), and $b: \mathcal{M} \times Q \to Q$ is an $\mathcal{M}$-Banach module action (the state-transition map). To each smooth linear system $M = (Q,f,I,g,Y,h)$ is naturally associated an $\mathcal{M}$-system, namely $\tilde{M} = (Q,\tilde{T},I,g,Y,h)$, where T is the semigroup generated by f. The association $M \mapsto \tilde{M}$ is injective, so $\mathcal{M}$-systems are indeed generalizations of smooth linear systems. $\mathcal{M}$-systems arising in this fashion from smooth linear systems will be termed <u>smooth</u>.

## 4. <u>Behavior of $\mathcal{M}$-Systems</u>

The universal construction scheme of defining behavior (outlined in section 2) which failed for smooth linear systems will now be shown to work for $\mathcal{M}$-systems, thus allowing smooth linear systems to be embedded into a behavioral framework. The constructions for free and cofree $\mathcal{M}$-Banach modules are similar to the classical cases over ordinary rings [6], but are repeated for clarity.

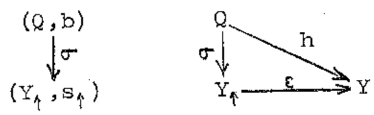
Given a Banach space I, a <u>free</u> $\mathcal{M}$-Banach module over I is an $\mathcal{M}$-Banach module $(I^\uparrow, s^\uparrow)$ and an $\eta \in BAN(I, I^\uparrow)$ such that for any $\mathcal{M}$-Banach module $(Q,b)$ and $g \in BAN(I,Q)$, there is a unique $\mathcal{M}$-Banach module morphism $\rho: (I^\uparrow, s^\uparrow) \to (Q,b)$ such the following diagram commutes.



$(I^\uparrow, s^\uparrow)$ exists for any Banach space I. Let E and F be Banach spaces. $E \otimes F$ denotes the tensor product of E and F, with $\|x\| = \inf\{\sum \|e_i\| \cdot \|f_i\| \mid x = \sum e_i \otimes f_i\}$ for $x \in E \otimes F$. This clearly turns $E \otimes F$ into a normed linear space; its completion is thus a Banach space which is denoted $E \hat{\otimes} F$.

THEOREM Given a Banach space I, $(\mathcal{M} \hat{\otimes} I, s^\uparrow)$ is a free $\mathcal{M}$-Banach module over I, with $s^\uparrow$ defined by $s^\uparrow(\mu, (\nu \otimes i)) = (\mu * \nu) \otimes i$, extended by $\mathbb{K}$ linearity and completion. $\eta: I \to I^\uparrow$ is the canonical injection $i \mapsto i \otimes \delta_0$. Given an $\mathcal{M}$-Banach module $(Q,b)$ and $g \in BAN(I,Q)$, $\rho: \mathcal{M}\hat{\otimes} I \to Q$ is defined by $\rho(\mu \otimes i) = b(\mu, g(i))$, extended by $\mathbb{K}$ linearity and completion.

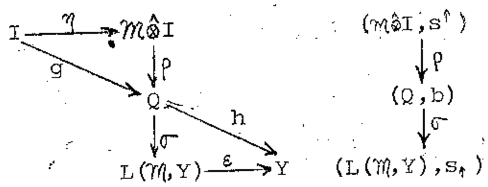
Given a Banach space Y, a <u>cofree</u> $\mathcal{M}$-Banach module over Y is an $\mathcal{M}$-Banach module $(Y_\uparrow, s_\uparrow)$ and an $\varepsilon \in BAN(Y_\uparrow, Y)$ such that for any $\mathcal{M}$-Banach module $(Q,b)$ and $h \in BAN(Q,Y)$, there is a unique $\mathcal{M}$-Banach module morphism $\sigma: (Q,b) \to (Y_\uparrow, s_\uparrow)$ such that the following diagram commutes.



$(Y_\uparrow, s_\uparrow)$ exists for any Banach space Y. Let E and F be Banach spaces, and let $L(E,F)$ denote the space of all continuous linear maps from E to F with norm $\|k\| = \sup\{\|k(e)\| \mid \|e\| \le 1\}$. It is well-known that $L(E,F)$ is a Banach space. Note that $L(E,F) \neq BAN(E,F)$, but rather $BAN(E,F)$ is the closed unit ball of $L(E,F)$.

THEOREM Given a Banach space Y, $(L(\mathcal{M},Y), s_\uparrow)$ is a cofree $\mathcal{M}$-Banach module over Y, with $s_\uparrow(\mu,k) = k(-*\mu)$. $\varepsilon: Y_\uparrow \to Y$ is the canonical evaluation $f \mapsto f(\delta_0)$. Given an $\mathcal{M}$-Banach module $(Q,b)$ and $h \in BAN(Q,Y)$; $\sigma: Q \to L(\mathcal{M},Y)$ is defined by $q \mapsto (\mu \mapsto h(b(\mu,q)))$.

Now the behavioral properties of $\mathcal{M}$-systems may be rigorously defined. Let $M = (Q,b,I,g,Y,h)$ be an $\mathcal{M}$-system. M gives rise to unique morphisms $\rho$ and $\sigma$ defined by the diagram below.



$\rho$ is the <u>reachability map</u>, $\sigma$ the <u>observability map</u>, and $\sigma \circ \rho$ the <u>behavior</u> of M.

Inputs over time to an $\mathcal{M}$-system with input space I are I-valued measures. Interpreting an element of $\mathcal{M}\hat{\otimes} I$ as an input, the time scale $\mathbb{R}_+$ is reversed to $\mathbb{R}_-$ under the natural association $t \mapsto -t$. Thus, the Dirac measure $\delta_t$ at $t \ge 0$ is interpreted as occurring at time $-t$, while an element $u \in L^1(\mathbb{R}_+) \hat{\otimes} I$ is interpreted as a function $u: \mathbb{R} \to I$. ( $L^1(\mathbb{R}_+)$, the space of absolutely Lebesgue integrable functions on $\mathbb{R}_+$, is a closed subspace of $\mathcal{M}$.) The reachability map $\rho: \mathcal{M}\hat{\otimes} I \to Q$ is given the usual interpretation [1,10]; it maps an input over $]-\infty, 0]$ to the resulting state at time 0. The response at time $t > 0$ to an input $u \in \mathcal{M}\hat{\otimes} I$ is found by translating u t units to the left to get $u * \delta_t$, and then applying $\rho$. The notations $\mathcal{M}^-$ and $\mathcal{M}^- \hat{\otimes} I$ will be used when it is necessary to consider the time scale to be $]-\infty, 0]$ on these spaces.

Outputs over time of an $\mathcal{M}$-system are continuous linear functions $\mathcal{M} \to Y$. This is a very rich space which cannot be completely characterized here, but it includes at least all Lebesgue measurable functions on $\mathbb{R}_+$ with values in Y. As is the usual convention with such algebraic representations, $\sigma(q)$ denotes the response to the zero input with initial state q at time 0.

$\mathcal{M}$-systems are a larger class than smooth linear systems, and so do not have in general the smoothness properties required for differential equation descriptions. However, each $\mathcal{M}$-system does have a "smooth" part which behaves very nicely, as is now illustrated.

Let $(Q,b)$ be an $\mathcal{M}$-Banach module. The <u>translation semigroup</u> $T_b: \mathbb{R}_+ \to BAN(Q,Q)$ is defined by $t \mapsto (q \mapsto b(\delta_t, q))$. In general, $T_b$ is not continuous (unless $(Q,b)$ is associated with a $(C_0)$ semigroup in the sense of section 3); however, there is a $(C_0)$ semigroup which can be extracted from it. Define $Q_b = \{q \in Q \mid t \mapsto b(\delta_t, q)$ is continuous $\mathbb{R}_+ \to E\}$. It is easy to see that $Q_b$ is a closed linear subspace of Q, hence a Banach space, and that $T_b$ maps $Q_b$ into itself. Let $S_b: \mathbb{R}_+ \to BAN(Q_b, Q_b)$ be the restriction of $T_b$ to $Q_b$.

THEOREM Let $(Q,b)$ be an $\mathcal{M}$-Banach module. Then $(Q_b, S_b)$ is a $(C_0)$ semigroup.

$(Q_b, S_b)$ is called the $(C_0)$ semigroup of $(Q,b)$; its infinitesimal generator is denoted by $\mathcal{G}_b$, rather than by the more cumbersome $\mathcal{G}_{S_b}$. Note that in a sense $(Q_b, S_b)$ is the largest $(C_0)$ semigroup contained in $(Q,b)$.

Several important examples of $(C_0)$ semigroups of $\mathcal{M}$-Banach modules are next presented.

THEOREM (a) Let $(\mathcal{M}^-, *)$ denote the $\mathcal{M}$-Banach module $\mathcal{M}$ itself, with time reversed to represent the inputs over time. Then $\mathcal{M}_*^- = L^1(\mathbb{R}_-)$, the space of absolutely (Lebesgue) integrable functions on $\mathbb{R}_-$. Translation is in the sense of measures and not of functions. Thus $S_*(t)(f) = g$, where $g(x) = f(x+t)$ for $x \le -t$, and $g(x) = 0$ for $x > -t$. The domain of $\mathcal{G}_*$ is the subspace of $L^1(\mathbb{R}_-)$ consisting of those differentiable functions which vanish at 0 (more precisely, those equivalence classes of functions which contain such a representative). $g_*$ is just differentiation.

(b) Given a Banach space I, $(\mathcal{M}^- \hat{\otimes} I)_s = L^1(\mathbb{R}_-) \hat{\otimes} I = L^1(\mathbb{R}_-, I)$, the space of all absolutely integrable functions with values in I. The $(C_o)$ semigroup structure is the natural extension of that of (a).

THEOREM Given a Banach space Y, $L(\mathcal{M}, Y)_{s_*} = \mathcal{B}(\mathbb{R}_+; Y)$, the Banach space of all uniformly continuous bounded functions $\mathbb{R}_+ \to Y$, normed by $f \mapsto \sup\{\|f(t)\| \mid t \in \mathbb{R}_+\}$. Translation is in the sense of functions, with $S_{s_*}(t)(f) = g$, where $g(x) = f(t+x)$. The domain of $g_s$ is the set of all such functions which are differentiable.

The $(C_o)$ semigroup part of an $\mathcal{M}$-Banach module is preserved under morphic image. This gives all $\mathcal{M}$-systems a degree of smoothness, which is detailed below.

THEOREM Let $k: (Q,b) \to (R,c)$ be an $\mathcal{M}$-Banach module morphism. Then $k(Q_b) \subseteq (R_c)$; i.e., k is also a morphism of the underlying semigroups.

Given Banach spaces I and Y, define an $\mathcal{M}$-system behavior to be any $\mathcal{M}$-Banach module morphism $B: (\mathcal{M} \hat{\otimes} I, s\uparrow) \to (L(\mathcal{M}, Y), s_*)$. By abuse of notation, the underlying BAN morphism $B: \mathcal{M} \hat{\otimes} I \to L(\mathcal{M}, Y)$ will also be termed a behavior.

COROLLARY Given an $\mathcal{M}$-system behavior $B: \mathcal{M} \hat{\otimes} I \to L(\mathcal{M}, Y)$, $B(L^1(\mathbb{R}_-, I)) \subseteq \mathcal{B}(\mathbb{R}_+, Y)$; i.e., the natural response (after the input has ceased) of any $\mathcal{M}$-system to an $L^1$ input is uniformly continuous.

This suggests that the dynamics of $\mathcal{M}$-systems may be described by differential and/or integral equations under suitable conditions. Results along these lines are recorded below.

THEOREM Let $M = (Q,b,I,g,Y,h)$ be an $\mathcal{M}$-system.

(a) For any input $i \in \mathcal{M} \hat{\otimes} I$, the state $q(t)$ at time $t \geq 0$ satisfies

$$q(t) = T_b(t)\rho(i).$$

(b) For any input $i \in L^1(\mathbb{R}_-, I)$, the system state $q(t)$ at any time t for which $i(t) \in \text{Domain}(g_b)$ satisfies

$$\frac{dq(t)}{dt} = g_b(q(t)) + g(i(t)).$$

(c) For any input $i \in L^1(\mathbb{R}_-, I)$ and $v \in \mathbb{R}_-$, let $i_o \in L^1(\mathbb{R}_-, I)$ be defined by $i_o(x) = i(x)$ for $x \leq v$, $i_o(x) = 0$ for $x > v$. Then the system state $q(t)$ at time $t \geq 0$ is given by

$$q(t) = T_b(t-v)\rho(i_o) + \int_v^t T_b(t-s)g(i(s))ds$$

provided that $i(t) \in Q_b$ for $t \geq v$.

Of course, whenever the state $q(t)$ is well-defined, the output is $h(q(t))$.

Even if the differential equation representation does not hold, it is still the case that a reasonably nice input will produce a smooth output, even while the input is being applied. This is summarized in the following.

THEOREM Let $M = (q,b,I,g,Y,h)$ be an $\mathcal{M}$-system. Let $u: \mathbb{R} \to I$ be any function such that $u^t: \mathbb{R} \to I$ defined by $u^t(x) = u(x+t)$ for $x \leq 0$ and $u^t(x) = 0$ for $x > 0$ is in $L^1(\mathbb{R}_-, I)$ for each $t \in \mathbb{R}$. Then the function $\mathbb{R} \to L^1(\mathbb{R}_-, I) : t \mapsto u^t$ is continuous when $L^1(\mathbb{R}_-, I)$ carries the topology inherited from $\mathcal{M} \hat{\otimes} I$.

Now regarding $\sigma \cdot \rho(u^t)$ as the output at time t of the system M in response to input u, the above theorem yields the following.

COROLLARY The response over time of an $\mathcal{M}$-system due to an $L^1$ input is continuous, even while the input is being applied.

## 5. Canonical Realization of $\mathcal{M}$-Systems

Let $B: \mathcal{M} \hat{\otimes} I \to L(\mathcal{M}, Y)$ be a behavior. A realization of B is an $\mathcal{M}$-system whose behavior is B. In classical finite-dimensional linear system theory, a system M is termed reachable if its reachability map $\rho$ is surjective, and observable if its observability map $\sigma$ is injective; M is termed canonical if it is both reachable and observable. The fundamental realization theorem for such systems then states that canonical realizations exist and are unique up to isomorphism. For many classes of infinite dimensional systems, the result in this exact form is no longer valid, and various alternatives have been considered to overcome this problem. Early definitions [4,11] required an (open surjection, injection) type factorization of the behavior to guarantee unique canonical realizations, while more recently Yamamoto [12] introduced the concept of topological observability, which requires a (dense maps, closed surjections) type of factorization of the behavior for unique canonical realizations. The categorical approach advanced here does not prefer any one of these definitions over the other, but rather bases the definition of canonical realization upon the categorical concept of image-factorization system, of which both of the above cases are examples (in appropriate categories). This approach was first advanced for discrete-time systems by Arbib and Manes [1,2], and later extended to certain classes of continuous-time systems by the author [7,8]. Essentially, an image-factorization system requires that each morphism have a factorization which is unique up to isomorphism. For completeness, the complete definition is repeated below.

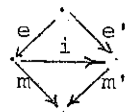
An image-factorization for a category $\mathcal{K}$ is a pair (E,M), where E and M are classes of $\mathcal{K}$ morphisms such that:

(i) both E and M are closed under composition;

(ii) $e \in E \Rightarrow e$ is an epimorphism;

(iii) $m \in M \Rightarrow m$ is a monomorphism;

(iv) i is an isomorphism $\Rightarrow i \in E \cap M$;
(v) each $\mathcal{K}$ morphism k has a factorization $m \circ e$ with $e \in E$ and $m \in M$ which is unique up to isomorphism in the sense that if $m' \circ e'$ is another such factorization, there is an isomorphism i such that the following diagram commutes.



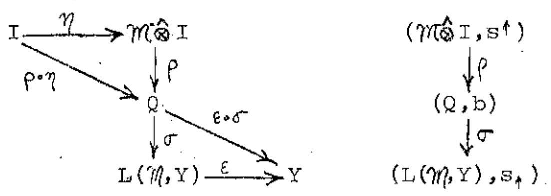

In the category of vector spaces over $\mathbb{K}$, (surjections, injections) is the only image-factorization system, so that factorization is in the usual sense. However, in categories of topologized vector spaces such as BAN or the category of locally convex spaces, this is no longer the case, and there are a variety of image-factorization systems. Two of the most useful for BAN are given below.

THEOREM (isometric surjections, injections) and (dense maps, isometric embeddings) are image-factorization systems for BAN.

Factorization of behaviors must take place in $\mathcal{M}$-MOD, and not in BAN; therefore, the above image-factorization systems must be extended to $\mathcal{M}$-MOD. Formally, an image-factorization system (E,M) for BAN lifts to $\mathcal{M}$-MOD if for each $\mathcal{M}$-MOD morphism $k: (P,a) \to (R,c)$ with $\bar{k}: P \xrightarrow{e} Q \xrightarrow{m} R$ an (E,M) factorization of $k \in BAN(P,R)$, there is a unique $\mathcal{M}$-Banach module structure b on Q such that $e: (P,a) \to (Q,b)$ and $m: (Q,b) \to (R,c)$ are $\mathcal{M}$-Banach module morphisms.

THEOREM Both (isometric surjections, injections) and (dense maps, isometric embeddings) lift to $\mathcal{M}$-MOD.

A formal definition of canonical realization may now be made. Let I and Y be Banach spaces, $B: (\mathcal{M}\hat{\otimes} I; s^{\dagger}) \to (L(\mathcal{M},Y), s_{\dagger})$ a behavior, and (E,M) an image-factorization system for BAN. The (E,M)-canonical realization of B is given as $M_B = (Q, b, I, \rho \circ \eta, Y, \varepsilon \circ \sigma)$, where $(\mathcal{M}\hat{\otimes} I, s^{\dagger}) \xrightarrow{\rho} (Q,b) \xrightarrow{\sigma} (L(\mathcal{M},Y), s_{\dagger})$ is a lifted (E,M) factorization of B; the definitions of the other maps are revealed by the following commutative diagram.




The above discussion shows that the following realization theorem is valid.

THEOREM (E,M) canonical realizations of behaviors exist and are unique up to isomorphism, for (E,M) = (isometric surjections, injections) and (dense maps, isometric embeddings).

A behavior $B: \mathcal{M}\hat{\otimes} I \to L(\mathcal{M},Y)$ is termed smooth if $B(\mathcal{M}\hat{\otimes} I) \subseteq \mathcal{B}(R_+,Y)$, i.e., if the natural response to each input is uniformly continuous.

THEOREM Given Banach spaces I and Y, and a behavior $B: \mathcal{M}\hat{\otimes} I \to L(\mathcal{M},Y)$, the following are equivalent.
(a) B is smooth.
(b) $B(\delta_0 \otimes i) \in \mathcal{B}(R_+,Y)$ for all $i \in I$.
(c) B admits a smooth realization (i.e., B is the behavior of a smooth linear system).
(d) The (dense maps, isometric embeddings) realization of B is smooth.

## 6. Conclusions

A behavior and realization theory for a class of systems called $\mathcal{M}$-systems has been presented. Smooth continuous-time linear systems defined over Banach spaces are a special case of $\mathcal{M}$-systems, so a realization theory for smooth systems has also been obtained. Furthermore, $\mathcal{M}$-systems have smooth responses to $L^1$ functions, and each $\mathcal{M}$-system has a part which is smooth enough to admit a differential equation model. A realization theory for $\mathcal{M}$-systems, together with criteria for a behavior to have a smooth realization, have been developed also.

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