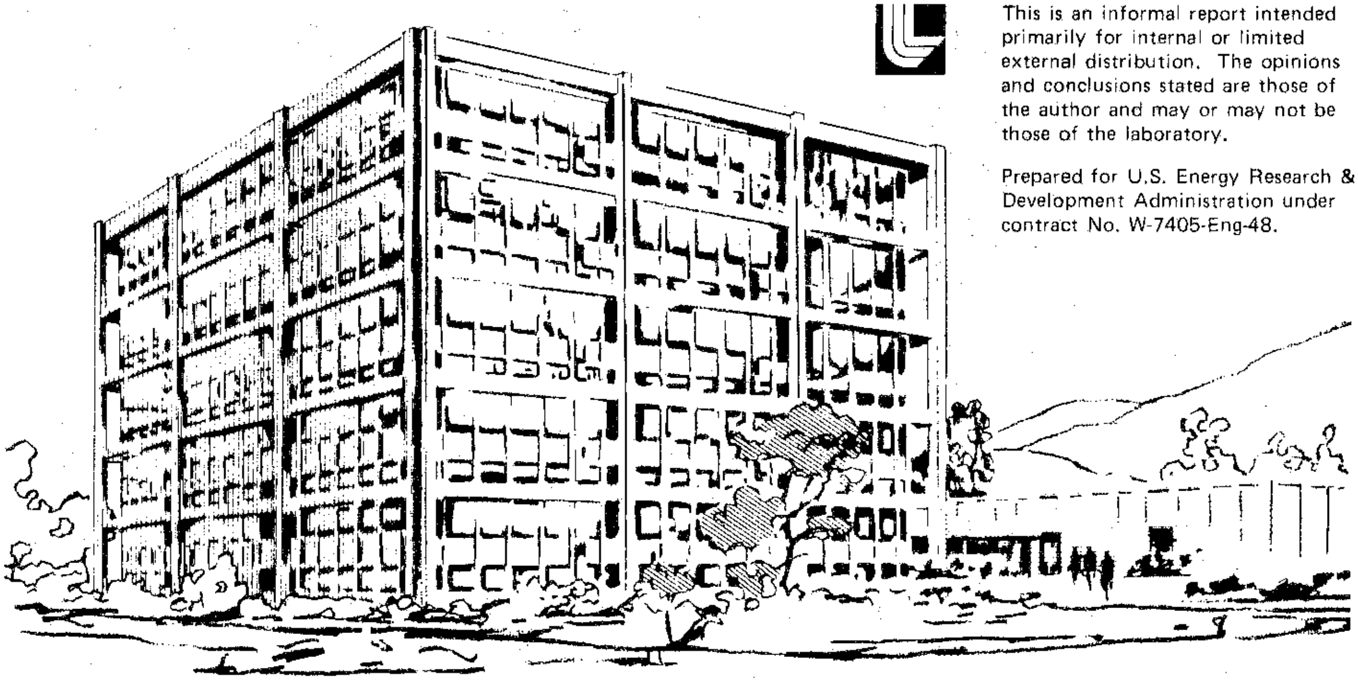
UCID-17760

Lawrence Livermore Laboratory

ALGEBRAIC SIMPLIFICATION OF INTERCONNECTED SYSTEMS

Stephen J. Hegner

March 31, 1978



Prepared for U.S. Energy Research & Development Administration under contract No. W-7405-Eng-48.

Algebraic Simplification of Interconnected Systems

by

Stephen J. Hegner

This work was prepared as a milestone report for the contract "Mathematical Theory of Modeling," funded by the U. S. Department of Energy, Division of Electric Energy Systems. It is meant to be self-contained and does not require, as a prerequisite, knowledge of other reports written for this contract. Nonetheless, there are many interesting connections with these other works. A preface, written by Dr. G. C. Corynen, thoroughly explores these connections, and in particular places the more specialized results presented herein within the more general theory of modelling and simulation developed by other investigators on this contract.

ABSTRACT

The overall topic of the report is the development of strategies for simplification of large scale interconnected systems by substituting, in the model, components which are simpler than the original ones. There are two main parts to the report. In the first, a general framework for discussing interconnection and interconnective modelling is developed, based upon the concepts of universal algebra. In the second part, the simplification of systems which may be described using matrices over a semiring is investigated in detail. Examples of such systems include the behaviors of multi-input multi-output linear systems.

Preface

SYSTEM STRUCTURE AND MODELING THEORY

G. C. Corynen

An important modeling question which arises in the study of large-scale systems may be roughly expressed as follows. Consider a system s and another system s' whose parts and interconnections model corresponding parts and interconnections of the original system s, to what extent is the whole system s' a model of s?

The answer to this question clearly depends upon what is meant by "model". The following paper reports work done on a contract whose principal purpose is to develop a mathematical theory of modeling which is based on a precise definition of the term "model". Two basic definitions have been developed in concurrent work(1), an abstract definition and a purposeful definition. While the former treats a modeling relation in some "absolute" sense, the latter explicitly accounts for the purposes for which models are developed.

Only the abstract definition is relevant to the paper presented here, and it is briefly reviewed to show the connection of this paper to the modeling theory under development.

Our modeling paradigm is illustrated in Figure 1. The sets S and S' are the original or prototype objects and model candidates, respectively. The maps Φ and Φ' are performance or feature extraction maps which associate, with each $s \in S$ and $s' \in S'$, a feature or performance $v = \Phi(s) \in V$ and $v' = \Phi'(s') \in V'$, respectively. These performance or feature evaluation maps allow the modeler to express the aspects or features of the prototypes and models which are important to him.

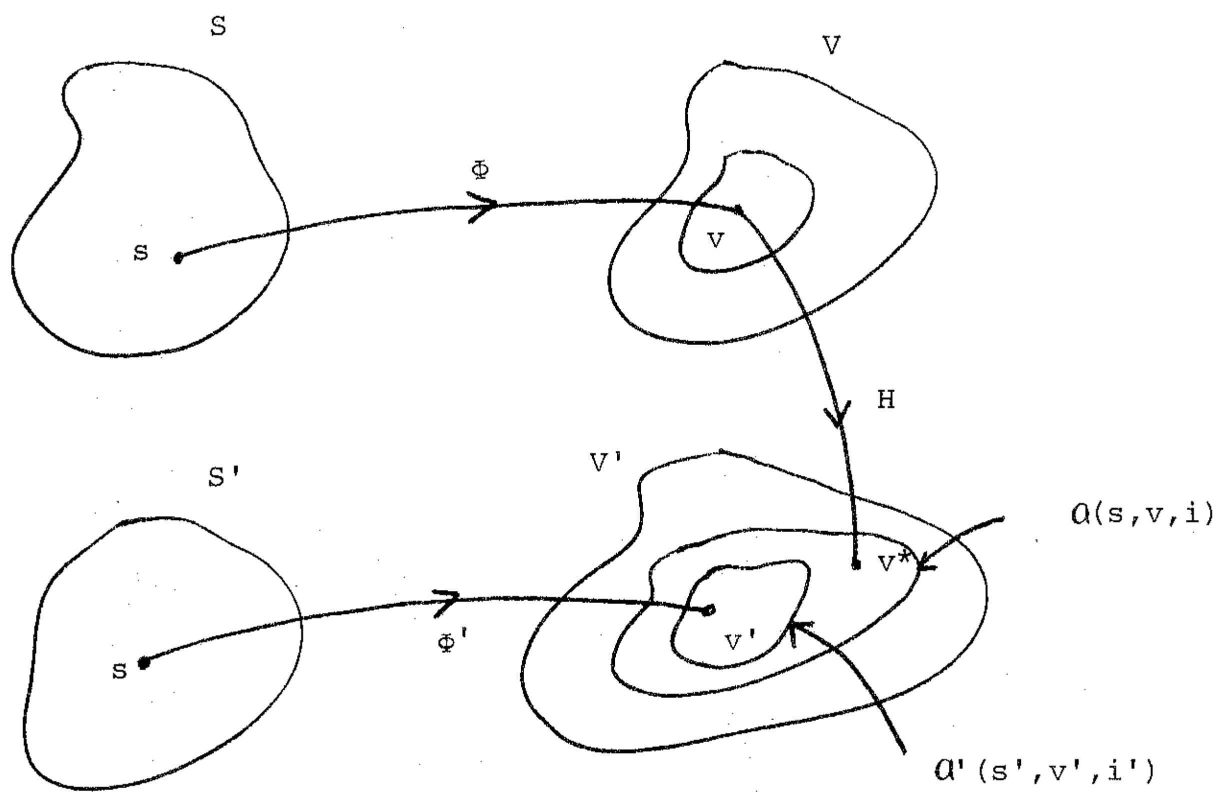


Figure 1. The important elements associated with the concept of "model".

Associated with these objects, maps and feature spaces we have various types of errors, uncertainties, ambiguities, tolerances and approximations. To capture these aspects, we introduce <u>approximation structures</u> which will be formally defined when the concept of "model is presented.

Finally, we have an interpretation relation $H \subset V' \times V$. If we are dealing with an activity where models are used as explanatory or predictive devices, H is used to make inferences in V about the prototypes from model features or outputs in V'. We call this the model use activity. If we are dealing with the process of modeling itself, where simplified or otherwise transformed objects and features are obtained from the prototypes and their features, then H is used to associate features in V' with the original features in V. This is called the model development activity. In both cases, the satisfaction of a modeling relation is determined by what the modeler wants from his models compared with what he actually gets from them. The formal definition of "model" is slightly different for each case and, because the following paper deals exclusively with model development, we summarize only the definition relevant to that activity. First, we introduce what we mean by approximation structures.

Definition In a model development activity, the prototype approximation structure (PAS) is a mapping

$a: S \times V \times I \rightarrow 2^{V'}$

where: S is the set of prototype objects

V is the set of prototype performances or features

I is a totally ordered set $<|I|, \alpha_I>$ with elements $|I|$ and total order[2] α_I on $|I|$.

$2^{V'}$ is the power set of V', with V' the model candidate performances or features.

This mapping may be interpreted as follows. If $v \in V$ is the feature of $s \in S$, $\mathcal{A}(s,v,i)$ is the set of model features which are approximations of v at level $i \in I$. The basic idea is that i reflects the modeler's standard of "closeness" between the model and the original object, and if the model candidate meets or exceeds this standard, the candidate qualifies as a model at "tolerance level i".

A similar structure is associated with the model candidates.

Definition For a model development activity, the model approximation structure (MAS) is a mapping $\mathcal{A}': S' \times V' \times I' \rightarrow 2^{V'}$

where S' is the set of model candidates

V' is the set of model features

I' is a totally ordered set $<|I'|, \alpha_I'>$

The usual interpretation of $\mathcal{a}$ and $\mathcal{a}'$ will be that $\mathcal{a}(s,v,i)$ are the <u>approximates tolerated at level i</u> and $\mathcal{a}'(s',v',i')$ are the <u>approximates obtained at level i'</u>.

In the main report [1], it is shown that this approximation framework captures topological, statistical, fuzzy, and combined notions of closeness and confidence in a natural way.

Let us now introduce the notion of "model". It is clear from the previous arguments that, whether some $s' \epsilon S'$ is a model of $s \epsilon S$ is not only a matter of degree but also depends on the modeler's point of view. This point of view is captured by introducing a <u>modeling criterion</u> $\mathcal{C} = <\Phi, \Phi', \mathcal{a}, \mathcal{a}'>$, where all symbols have been defined above. The modeler is then represented by his <u>point of view</u> $\mathcal{M} = <\mathcal{C}, H>$, where H is his interpretation. We call $\mathcal{M}$ itself <u>the modeler</u>.

<u>Definition (Model)</u> Given a modeler $\mathcal{M} = <\mathcal{C}, H>$, tolerance and obtainment levels i and i', respectively, (Figure 1), then <u>$s' \epsilon S'$ in a model of $s \epsilon S$ at tolerance level i with strength i' relative to $\mathcal{M}$,</u> if and only if

$$\mathcal{a}'(s', \Phi'(s'), i') \subset \mathcal{a}(s, \Phi(s), i)$$

We denote this condition by $s' m(\mathcal{M}; i, i') s$.

The following paper discusses the modeling development process where a complex system is modeled "piecemeal", by modeling each of its parts and their interconnections, and it is asked: under what conditions is the resulting interconnection of models a model of the original system?

More specifically, this paper considers the case where $\Phi: S \rightarrow V$ extracts the input/output (I/O) characteristics (behaviors) of interconnected systems in S. Similarly, S' are interconnections of simpler components, V' is a set of simplified performances, the I/O connectivities, and Φ' associates with each simplified model s' its connectivity in V'. It is assumed that $H: V \rightarrow V'$ is a function and that no approximation is associated with the model candidates and that an ordering $\alpha_{V'}$ (a complexity ordering, for instance) is given on V' where v_2' is preferred to v_1' if and only if $v_1' \alpha_{V'} v_2'$. The interpretation of this as a modeling problem is immediate: s' is a model of s if and only if $\Phi'(s')$ is preferred to H(v), where v is the feature of the original system. In our modeling framework, $I' = I = \{0,1\}$ and

$$\forall s \in S,\ \mathcal{a}(s,v,i) = \begin{cases} \{v' \in V' : H(v)\alpha_{V'} v'\} = \alpha_{v'}, & i = 0 \\ V', & i = 1 \end{cases}$$

$$\forall s' \in S',\ \mathcal{a}'(s',v',i') = \begin{cases} \{v'\}, & i = 0 \\ V', & i = 1 \end{cases} .$$

Only models with tolerance level zero and strength zero are thus considered, and

$$s' m(\mathcal{M};0,0)s \text{ iff } \{\Phi'(s')\} \subset A_{V'}$$

or

$$s' m(\mathcal{M};0,0)s \text{ iff } H \circ \Phi(s)\ \alpha_{V'}\ \Phi'(s') .$$

References

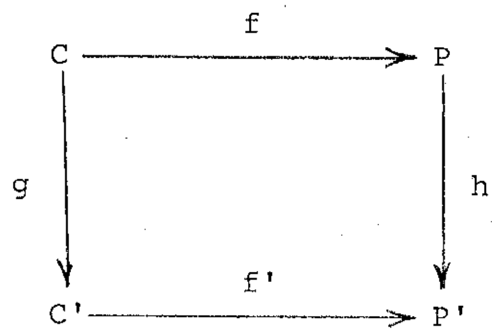
(1) G. C. Corynen: *A New Look at the Concept of Model: A Compatibility Principle,* report in progress.

(2) J. Dugundji: *Topology,* Allyn and Bacon, 1966.

1. INTRODUCTION

Large-scale systems may frequently be viewed as interconnections of relatively simple component systems. While a precise analysis of each component system may be entirely feasible, as may the analysis of an interconnection of a relatively small number of such components, the sheer number of such components in a typical large-scale system may preclude the computation of a property such as overall system behavior. Therefore, it is reasonable to look for techniques which simplify such large-scale systems to the point where computations become feasible, yet enough information is retained to allow useful analysis. One such technique is to substitute for each component in the original system a simpler component. This component must of course retain useful information about the original component. The following question may then be asked: What information about the original interconnected system may be obtained by investigating the interconnection of simplified components?

More completely, suppose C is the set of component from which the original system is built. Given is a set P of performances (e.g., behaviors) and a function $f:C \rightarrow P$ assigning a performance to each component. Given also is a set C' of simpler components, a set of simplified performances P' (e.g., connectivity between inputs and outputs), and a function $f':C' \rightarrow P$ assigning a performance to each simplified component. Now suppose that a map $g:C \rightarrow C'$ specifies the substitution of components in C by simpler components in C'. This substitition respects the performances if there is a map $h:P \rightarrow P'$ such that



commutes. Note that h is unique on f(C), if it exists. It might be said that g is an exact modelling substitution relative to (f,f').

Given such a modelling-of-components relationship, it is relevant to ask what it implies about modelling interconnected systems, assuming that g specifies how components are to be substituted. In this paper, the mathematical machinery necessary to precisely address such a query is developed. In the most general framework, developed in sections 2 and 3, the components are allowed to be completely abstract. The key mathematical concept turns out to be universal algebra; it allows the above diagram to be "lifted" to the world of interconnected systems. While the full commutativity may be lost, the somewhat weaker but nonetheless useful concept of semi-commutativity qualifies the modelling relationship which is obtained.

In sections 4 through 6, the framework of sections 2 and 3 is used to illustrate interconnective modelling in the case in which the components are matrices defined over a semiring, and the interconnection operators are simple operations on these matricies. This specialized framework recaptures in particular the interconnection theory of behaviors of multi-input, multi-output linear systems. The computation of the resulting behavior of the interconnection of a large number of such systems is known to be computationally complex. Several useful simplification techniques, most of which use only binary data, are illustrated. The computation of the performance of the interconnections of these simplified models is are less complex. Nonetheless, they all carry useful information, such as connectivity, length of delay, etc., thus illustrating the utility of the abstract theory.

2. ALGEBRAIC INTERCONNECTION SPACES

In the most basic type of interconnection theory, a set of systems A is given, together with a set of rules S for combining systems in A to form new systems. The elements in S are thus functions of the form

$$f: A^n \to A$$

However, not all systems are interconnectable in all possible ways, in general. That is, the function f above is in general a partial function. For example, if A is a set of systems with multiple inputs and multiple outputs and $g: A^2 \to A$ is series interconnection, then g is a partial function which is defined only if the number of output lines of the first component is the same as the number of input lines of the second component. Thus, g is partial, but in a nice way. This suggests that interconnections may be characterized by certain types of partial functions. That this is so is now illustrated.

Let A and M be sets. An M-typing of A is a function $X: M \to 2^A$. Such an f is also called an M-set, and M is called a set of types. The idea is that each element of A is assigned zero or more types from M; $a \in A$ has type $m \in M$ if $a \in X(m)$. As a simple example, if A is a family of multiple-input multiple-output systems, each $a \in A$ has one type from $M = \underline{\underline{N}}\ \underline{\underline{N}}$ (where $\underline{\underline{N}}$ = natural numbers), namely $a \in X((m,n))$ means that a has m input lines and n output lines.

The partial function $f:A^n \rightarrow A$ in the example above thus may be replaced by a total function of the form $f:X(m_1)\times \ldots \times X(m_n) \rightarrow A$, where $X:M \rightarrow 2^A$ is an appropriate M-function on A, and $m_1,\ldots,m_n \in M$. However, it is usually the case that the interconnected system will also have a definite type $m_{n+1} \in M$. Thus $f:X(m_1)\times \ldots \times X(m_n) \rightarrow X(m_{n+1})$ is an even more appropriate form for an interconnection operator.

The mathematical concept which permits a systematic treatment of interconnections of systems is (heterogeneous) universal algebra. The main ideas of this theory are now recalled. For a more complete discussion, consult [Hegner, 1977]. For a general discussion of hetergeneous algebras, consult [Birkhoff and Lipson, 1970].

Let M be a set, the set of types. Let M^+ denote the set of all finite nonempty sequences of elements from M. An M-operator domain is any M^+-function Ω such that $\Omega(m) \cap \Omega(m') \neq \emptyset \Rightarrow m=m'$. An Ω-algebra is a pair (X,δ) where X is an M-set and δ is a map which assigns to each $m_1 \ldots m_n \in M^+$ and $\omega \in \Omega(m_1 \ldots m_n)$ a function $\delta_\omega : X(m_1)\times \ldots \times X(m_{n-1}) \rightarrow X(m_n)$. The M-set X is called the carrier of the algebra.

Returning to the previous example, if X is a set of multiple-input multiple-output systems, $M = \underline{\underline{N}} \times \underline{\underline{N}}$, and $x((m,n))$ is the set of systems with m inputs and n outputs, then the series interconnection operator of m-input n-output systems with n-input p-output systems is labelled by an element ω of $\Omega((m,n)(m,p),(m,p))$. That is $\delta_\omega : X((m,n)) \times X((n,p)) \rightarrow X((m,p))$.

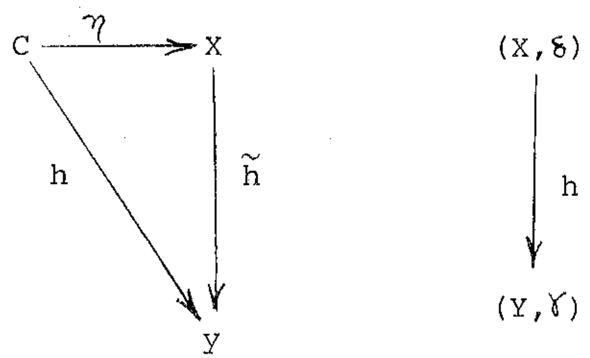
To describe relationships between different interconnection schemata and hence different algebras, it is necessary to introduce the concept of morphism of algebras. This requires the notion of M-set morphism. Let M, A, B be sets, and let $X: M \rightarrow 2^A$ and $Y: M \rightarrow 2^B$ be M-sets. A morphism form X to Y (called an M-<u>function</u>) is an M-set $f: M \rightarrow 2^{[2^A \rightarrow 2^B]}$ such that for each $m \in M$, $f(m)$ maps $X(m)$ into $Y(m)$; $f: X \rightarrow Y$ is sometimes written. Now let Ω be an M-operator domain, and let (X,δ) and (Y,γ) be Ω-algebras. An Ω-<u>algebra</u> <u>morphism</u> from (X,δ) to (Y,γ) is an M-function $f: X \rightarrow Y$ such that for each $m_{.} \ldots m_n \in M^+$ and $\omega \in \Omega(m_1 \ldots m_n)$, the diagram

$$\begin{array}{ccc}
X(m_1) \times \ldots \times X(m_{n-1}) & \xrightarrow{\delta_\omega} & X(m_n) \\
{\scriptstyle f(m_1) \times \ldots \times f(m_{n-1})} \downarrow & & \downarrow {\scriptstyle f(m_n)} \\
Y(m_1) \times \ldots \times Y(m_{n-1}) & \xrightarrow{\gamma_\omega} & Y(m_n)
\end{array}$$

commutes.

In a system-theoretic context, think of X and Y as M-sets of systems, and $f:X \rightarrow Y$ as a map which says "substitute f(m)(b) for b if b is of type $m \in M$." If $f:(X,\delta) \rightarrow (Y,\gamma)$ is a morphism, then systems may be first interconnected and then a substitution made for the result, or else each component system may be first be substituted for and then the substitutes interconnected. The result in each case is the same.

A particularly important concept is that of the free algebra. Let M be a set, Ω an M-operator domain, and C an M-set. A <u>free Ω-algebra</u> over C is a pair $(\eta,(X,\delta))$ where (X,δ) is an Ω-algebra and $\eta:C \rightarrow X$ is an M-function such that for any other such pair $(h,(Y,\gamma))$, there is a unique Ω-algebra morphism $h:(X,\delta) \rightarrow (Y,\gamma)$ such that



commutes.

An Ω-algebra morphism $f:(X,\delta)\to(Y,\gamma)$ is called an <u>isomorphism</u> if $f(m)$ is bijective for each $m \in M$.

THEOREM 2.1 Let C be an M-set, Ω an M-operator domain. Free Ω-algebras over C are unique up to isomorphism in the sense that if $(\eta,(X,\delta))$ and $(\eta'(Y,\gamma))$ and each free Ω algebras over C, there is an Ω-algebra isomorphism $f:(X,\delta)\to(Y,\gamma)$.

Equal in importance to the uniqueness of free Ω-algebras is their existence. Fortunately, they always exist. Let M be a set, C an M-set, and Ω an M-operator domain. The Ω-<u>terms</u> of type $m \in M$ over C are defined to be those objects obtainable from the following two rules in a finite number of steps.

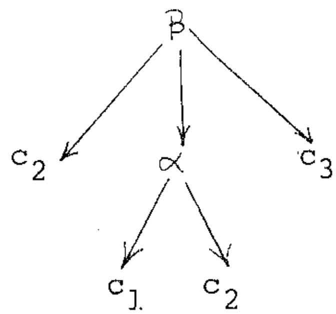
(i) If $c \in C(m)$, then c is an Ω-term over C of type m. (These are called <u>atomic</u> Ω-terms of type m.)

(ii) If $m_1 \ldots m_n \in M^+$, $m_n=m$, c_i is an Ω-term of type m_i for $1 \leq i \leq n-1$, and $\omega \in \Omega(t_1 \ldots t_n)$, then the string $\omega(c_1,\ldots,c_n)$ is an Ω-term over C of type m. (These are called <u>nonatomic</u> Ω-terms of type m.).

The M-set of all Ω-terms over C is denoted ΩC. ΩC is the carrier of an Ω-algebra $(\Omega C,1)$ in a natural way. If $m = m_1 \ldots m_n \in M^+$, $\omega \in \Omega(m)$, and $c_i \in (\Omega C)(m_i)$ for

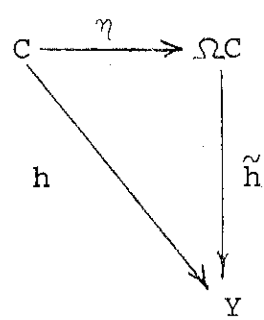
$1 \leq i \leq n-1$, $\omega(c_1, \ldots, c_{n-1})$ is the element of type m_n in ΩC which is just that as a formal string of symbols. This algebra is called the Ω-<u>term algebra</u> over C.

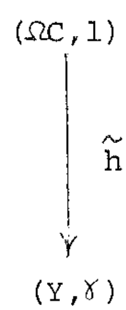
An Ω-term may be thought of as a tree whose decendants are ordered. For example, the Ω-term $\beta(c_2, \alpha(c_1, c_2), c_3)$ represented by the tree



THEOREM 2.2 Let M be a set, C an M-set, and Ω an M-operator domain. Define $\eta: C \to \Omega C$ to be the M function which is the inclusion map for each $m \in M$. Then $(\eta, (\Omega C, 1))$ is a free Ω algebra over C.

Given an Ω-algebra (Y, γ) and an m-function $h: C \to Y$, it is easy to see what the homorphism h below must be.

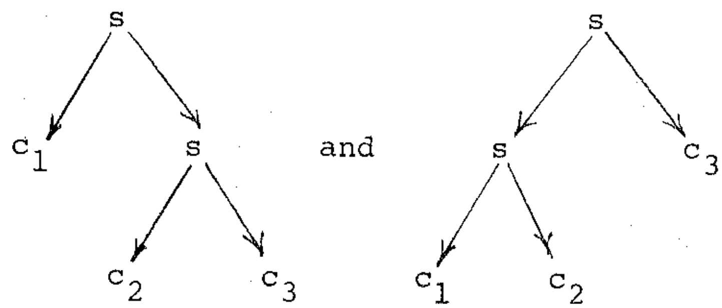




Namely, for $T \in \Omega C$, replace each occurrence of an element of C of type m with h(m)(c), and replace each occurrence of $\omega \in \Omega(m_1 \ldots m_n)$ with γ_ω. Then evaluate the resulting expression.

The Ω-term algebra has a very useful system-theoretic interpretation. Consider the M-set C as a set of basic or primitive components which are to be interconnected. The elements of C then represent all of the possible ways in which the components may be interconnected, i.e., the formal interconnections. The map h specifies the property of interest for each component. For example, Y may be a set of behaviors; then $h: C \to Y$ specifies the behavior of each component. The extension map $\tilde{h}: \Omega C \to Y$ then gives the behavior of each interconnected system. The algebra $(\Omega C, 1)$ is often called the <u>syntax</u> of the interconnection and (Y, γ) the <u>semantics</u>.

There is one small problem with the definition of syntax which is given above. ΩC captures not only how the components are interconnected, but also in what order they were interconnected. For example, if the components c_1, c_2, and c_3 are connected in series with s, the series interconnection operator of two components, then the interconnections represented by



each describe this situation. The one on the left says that c_2 and c_3 were first connected in series and then c_1 was connected in series with the result. The one on the right says that c_1 is first interconnected in series with c_2, and this result is then put into series with c_3. In most cases, it is not necessary or desirable to distinguish between these two cases, and some method for declaring them equivalent would be useful. Such a method is available; it is the method of equational classes. It will now be briefly described.

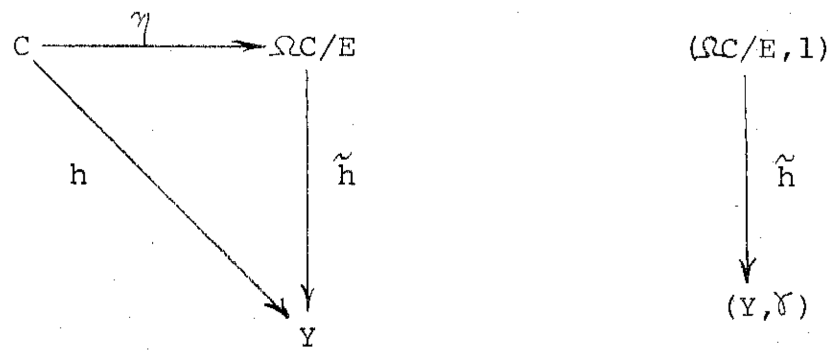
Let M be a set and Ω an M-operator domain. Let $V = \{v_0, v_1, v_2, \ldots\}$ be an effectively enumerated set of variables. The M-set of V, denoted V_M, is given by $V_M : M \to M \times V$, $m \leadsto (m, v)$. An Ω-<u>equation</u> is a pair of Ω-terms over V_M which are of the same type. An M-set of Ω-equations is an M-set E such that E(m) is a set of Ω-equations of type m.

Let (X,δ) be an Ω-algebra and let E by an M-set of Ω-equations. (X,δ) satisfies E if for every M-function $h:V_M \to X$ and every $\{e_1,e_2\} \in E(m)$, the unique extension $h:(\Omega V_m,1) \to (X,\delta)$ satisfies $h(m)(e_1) = h(m)(e_2)$. If (X,δ) satisfies E it is called an (Ω,E)-algebra.

Free (Ω,E) algebras over C are defined just as free Ω-algebras are. In the definition, simply replace each occurrence of the word "Ω-algebra" with the word "(Ω,E)-algebra". (Ω,E) algebras exist and are unique up to isomorphism. There is a canonical free (Ω,E) algebra over any M-set C, which is defined as an appropriate quotient (homomorphic image) of the Ω-term algebra $(\Omega C,1)$. More precisely, define a relation $E_C(m)$ on $(\Omega C)(m)$ for each $m \in M$ by $(t_1,t_2) \in E_C(m)$ if for all (Ω,E)-algebras (Y,γ) and all M-functions $h:C \to Y$, $(h(m))(t_1) = (h(m))(t_2)$, where $h:(\Omega C,1) \to (Y,\gamma)$ is the induced algebra morphism. For each $m \in M$, $E_C(m)$ is clearly an equivalence relation. Let $\Omega C/E$ denote the M-set with $(\Omega C/E)(m) = (\Omega C)(m)/E_C(m)$. Let $p: C \to C/E$ denote the M-function which is the canonical projection on each $\Omega C(m)$. Write [t] for $p(m)(t)$. There is a natural Ω-algebra structure on $\Omega C/E$ given by $\omega([c_1],\ldots(c_n]) = [\omega(c_1,\ldots,c_n)]$. Denote this algebra $(\Omega C/E,1)$. It is the canonical free (Ω,E)-algebra.

In an interconnection framework, E specifies the basic interconnections which are designated to be equivalent. The algebra $(\Omega C/E,1)$ gives the syntax of the interconnections modulo these designated equivalences. The rest of the analysis is as before.

Let M be a set, Ω an M operator domain, and E an M set of Ω-equations. An (Ω,E)-algebraic interconnection space is an ordered triple $(c,(Y,\gamma),h)$, where C is an M-set, called the set of components, (Y,γ) is an (Ω,E)-algebra, called the semantics of the interconnection space, and $h:C\to Y$ is an M-function, called the semantics of components map. The unique Ω-algebra morphism $h:(\Omega C/E,1)\to(Y,\gamma)$ guaranteed by the freeness of $(\Omega C/E,1)$

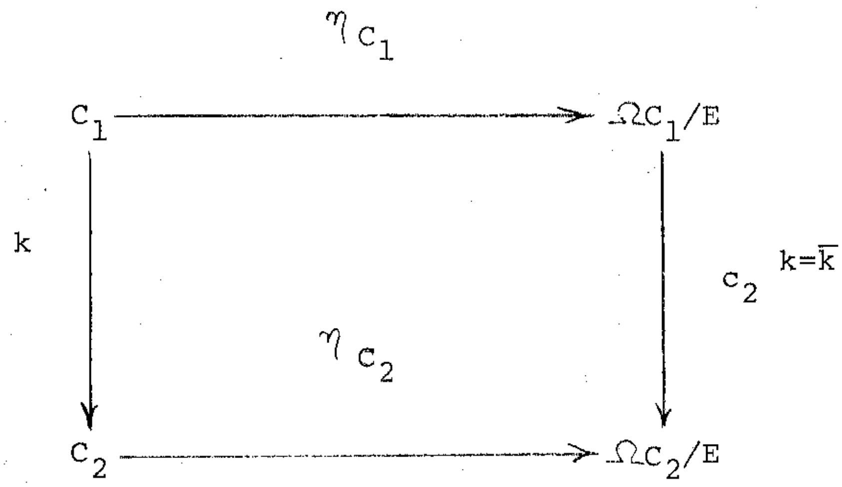


is called the semantics of systems map, The concept captured here is that by knowing only the semantics of components, the semantics of interconnected systems may be recovered.

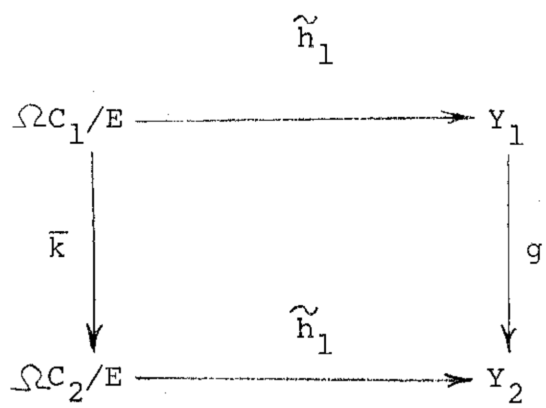
3. EXACT INTERCONNECTIVE MODELS AND SEMI-INTERCONNECTIVE MODELS

The problem of modeling in a interconnection framework is basically the following. There are two algebraic interconnection spaces present, the actual components space and the model components space. The underlying idea is that by substituting model components for actual components, the concept of interconnection model may be made precise. The idea of exact interconnective model, first present in [Hegner, 1977], will be reviewed first, the new and more liberal concept of semi-morphic interconnective model will then be discussed.

Fix a set M, M-operator domain Ω, and Ω-equations E. Let $S_i = (C_i,(Y_i,\gamma_i),h_i)$ for i = 1,2 be algebraic interconnection spaces. A <u>modeling specifier of</u> S_2 <u>by</u> S_1 is an M-function $k:C_1 \rightarrow C_2$. The intuitive idea is that for each component $c \in C_1(m)$, the component $k(m)(c) \in C_2(m)$ will be substituted. k induces an Ω-algebra morphism $\overline{k}:(\Omega C_1/E,1) \rightarrow (\Omega C_2 E,1)$ with $\overline{k} = \widetilde{\eta_{C_2} \circ k}$ as illustrated below.



The map $\bar{k}$ tells how entire interconnected systems in $\Omega C_2/E$ must be substituted for systems in $\Omega C_1/E$, given k. k is an <u>exact interconnective</u> modeling specifier if it respects the semantics, that is, if there exists an Ω-algebra morphism $g:(Y_1\ _1)\quad(Y_2,\ _2)$ such that the diagram



commutes. If g exists, it is uniquely defined on $\tilde{h}_1(\Omega C_1/E)$. If each component of h_1 is surjective, g is unique. g tells how to substitute for semantics. If the semantics of interconnected system s_1 is $h_1(m)(s_1)$, then the semantics of interconnected system $s_2 = \bar{k}(m)(s_1)$ must be $h_2(m)(s_2)$. The property of being exact interconnective insures that these semantics are morphically related.

There are, however, many examples occurring in practice, as illustrated later in this report, for which no such g exists, and so h is not exact interconnective. For this reason, other concepts of interconnective model have been developed. For example, [Aggarwal,1977] has developed the concept of composition space. Here another concept, that of semi-interconnective model, has been developed. Its utility will be demonstrated clearly in the examples in the next sections.

To deal with semi-interconnective models, the concept of algebras over partially-ordered sets must first be introduced. Let X be a set and $\leq$ a relation on X. $\leq$ is a <u>partial order</u> on X if it satisfies the following three rules:

(p.o. 1) It is <u>reflexive</u>, i.e., for all $x \in X$, $(x,x) \in \leq$.

(p.o. 2) It is <u>antisymmetric</u>, i.e., $(x,y) \in \leq$ and $(y,x) \in \leq \Rightarrow x = y$.

(p.o. 3) It is transitive, i.e., $(x,y) \in \leq$ and $(y,z) \in \leq \Rightarrow (x,z) \in \leq$.

$x \leq y$ is usually written for $(x,y) \in \leq$. $(X,\leq)$ is called a partially-ordered set. Sometimes X is called a partially-ordered set, when $\leq$ is understood. Let X and Y be partially-ordered sets. A function $f:X \to Y$ is a partially-ordered set morphism or istotonic function, if $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. Let $\{X_i | i \in I\}$ be a family of partially-ordered sets. The product of $\{X_i | i \in I\}$ has $\prod_{i \in I} X_i$ as its underlying set, with $x \leq y \Leftrightarrow x_i \leq y_i$ for all $i \in I$. Let M be a set, and let X by an M-set such that each X(m) has a partial order associated with it. Then X is called a partially ordered M-set. A morphism of partially-ordered M-sets is an M function, each of whose components is isotonic.

Let Ω be an M-operator domain, and let (X,δ) be an Ω-algebra. (X,δ) is a partially-ordered Ω-algebra if X is a partially-ordered M-set and for each $t \in M^+$ and each $\omega \in \Omega(t)$, δ_ω is isotonic. If (X,δ) and (Y,γ) are partially-ordered Ω-algebras and $f:(X,\delta) \to (Y,\gamma)$ is an Ω-algebra morphism, then f is a partially-ordered Ω-algebra morphism if f is a morphism of partially-ordered M-sets.

Now it can also be shown that free partially-ordered Ω- and (Ω,E)-algebras exist, and so the concept of exact interconnective model also makes sense in this more general framework. However, this would not really add anything new, and so the details will not be pursued here. Rather, a more liberal concept modeling. Namely, the semantics obtained by interconnecting the substituted components is always greater than the semantics obtained by evaluating the original components.

Let M, Ω, E, S_i, $i=1,2$, and $k: C_1 \to C_2$ be as in the beginning of this section. Assume further that (Y_2, γ_2) is a partially-ordered algebra. k is a <u>semi-interconnective</u> modeling specifier if there exists an Ω-algebra semimorphism $g: (Y_1, \gamma_1) \quad (Y_2, \gamma_2)$ such that the diagram

$$\begin{array}{ccc} \Omega C_1/E & \xrightarrow{\tilde{h}_1} & Y_1 \\ {\scriptstyle \bar{k}}\downarrow & & \downarrow{\scriptstyle g} \\ \Omega C_2/E & \xrightarrow{\tilde{h}_2} & Y_2 \end{array}$$

semicommutes. g is said to be compatible with k. g is <u>optimal</u> if, for all $x \in \tilde{h}_1(\Omega C_1/E)$, $f(x) \leq g(x)$ for all other Ω-algebra semimorphisms $f: Y_1 \to Y_2$ which makes the diagram above semicommute. Note that if $\tilde{h}_1$ is surjective, this reduces to $f \leq g$. (If $\tilde{h}_1$ is not surjective, the values of g on arguments not in $\tilde{h}_1(\Omega C_1/E)$ are of no consequence in the modeling relationship.)

of interconnective model based upon the concept of semi-morphism is presented.

Lex X be a set and let Y be a partially-ordered set. On Y^X, the set of all functions from X to Y, may be placed a natural partial order. Namely, $f \leq g$ if for all $x \in X$, $f(x) \leq g(x)$.

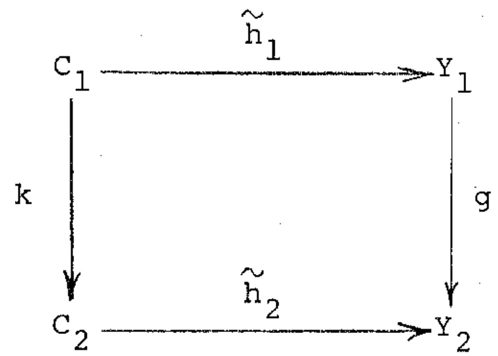
Let (X,δ) and (Y,γ) be Ω-algebras, and let $f:X \rightarrow Y$ be an M-function. Suppose that (Y,γ) is in addition a partially-ordered Ω-algebra. f is called a <u>semi-morphism</u> from (X,δ) to (Y,γ) if for each $m_1 \ldots m_n \in M^+$ and each $\omega \in \Omega(m_1 \ldots m_n)$, the diagram

$$\begin{array}{ccc} X(m)_1 \times \ldots \times X(m_{n-1}) & \xrightarrow{\delta_\omega} & X(m_n) \\ {\scriptstyle f(m_n) \times \ldots \times f(m_{n-1})} \downarrow & & \downarrow {\scriptstyle f(m_n)} \\ Y(m_1) \times \ldots \times Y(m_{n-1}) & \xrightarrow{\gamma_\omega} & Y(m_n) \end{array}$$

<u>semicommutes</u>, in the precise sense that $f(m_n) \circ \delta_\omega \leq \gamma_\omega \circ (f(m_1) \times \ldots \times f(m_{n-1}))$. That is, the "top then right" path is less than the "left then bottom" path.

The concept of semi-morphism is clearly weaker than the concept of morphism, since every morphism is a semi-morphism. However, it still conveys information regarding interconnective

THEOREM 3.1 Let M be a set of types, Ω an M operator domain, E a set of Ω equations, let (Y_i,γ_i) for i=1,2 be (Ω,E) algebras, and let S_i $(C_i,(Y_i,\gamma_i),h_i)$ be algebraic interconnection spaces. Suppose in addition that (Y_2,γ_2) is a partially-ordered Ω-algebra, and let $k:C_1 \rightarrow C_2$ be a modelling specifier. Let $g:Y_1 \rightarrow Y_2$ be compatible with k. For g to be optimal, it is sufficient that the diagram below commutes.



Proof: By the unique extension property of free algebras, the diagram above determines g uniquely on $\tilde{h}_1(\Omega C_1/E)$, i.e., g on $\Omega C_1/E$ is determined completely by g on $h_1(C_1)$. Clearly, commutativity is the strongest type of semicommutativity, hence g is optimal.

4. ALGEBRAIC OBJECTS

In many examples of system interconnection, the objects to be interconnected have natural algebraic operations associated with them. It is the purpose of this section to formalize these basic algebraic operations, as well as to give several useful examples.

The concepts of ring and of module over a ring are very useful ones in mathematics, and a great deal has been written about them. Consult any good text on algebra for a general discussion (e.g., [Birkhoff and MacLane, 1967], [Hungerford, 1974], [Lang, 1965]). However, for many of the examples naturally encountered in the study of large-scale systems, the axioms for a ring are a bit strong, and the weaker concept of semiring seems more appropriate. Semirings have been seen substantial use in theoretical computer science, and the reader is referred to texts such as [Eilenberg, 1974, 1976] for a detailed discussion. Finally, the usual concepts of morphisms, both for rings and semirings, are too strong for many of the examples encountered in large-scale systems. The approach taken here is to assign compatible partial orders to the semirings, and then introduce the weaker concept of semimorphism. A discussion of all these topics follows.

Monoids and Semirings

A monoid is a triple $(X,\cdot,1)$, where X is a set, $\cdot:X\times X \rightarrow X$ is an associative binary operation on X, and $1 \in X$ with $1\cdot x = x\cdot 1 = x$ for all x X. 1 is called the identity of the monoid. When there is no possibility of confusion, it will simply be said that X is a monoid. X is commutative if $x\cdot y = y\cdot x$ for all $x,y \in X$.

A commutative semiring with unit (hereafter just semiring) is a pair of commutative monoids $((S,+,0), (S,\cdot,1))$ subject to the following two laws:

(a) $x\cdot(y+z) = x\cdot y + x\cdot z$ for all $x,y,z \in S$.

(b) $x\cdot 0 = 0\cdot x = 0$ for all $x \in S$.

Again, when no confusion can result, it will simply be said that S is a semiring, with the above notational conventions. However, $x\cdot y$ will be written as just xy.

EXAMPLE 4.1 Let $\underline{\underline{N}}$ denote the natural numbers $\underline{\underline{Z}}$ the integers, $\underline{\underline{Q}}$ the rational numbers, $\underline{\underline{R}}$ the real numbers, $\underline{\underline{C}}$ the complex numbers. Each is a semiring under the usual operations. Any field or ring is a semiring.

EXAMPLE 4.2 Let S be any semiring, Let S[[x]] denote the set of all functions $f:\underline{\underline{N}}\rightarrow S$. It is convenient to write f as a formal power series $\sum_{k=0}^{\infty} a_k x^k$, with $a_k = f(k)$. Often

$\sum a_k X^k$ will be written. Define addition on S[[x]] pointwise, i.e., $\sum a_k x^k + \sum b_k x^k = \sum (a_k+b_k) X^k$. Multiplication is defined as <u>convolution</u>, and often written as *: $\sum a_k x^k + \sum b_k x^k = \sum c_k x^k$, where $c_k = \sum_{i+j=k} a_i b_j$. The series 1 with $a_0=1$, $a_k=0$ for $k > 0$ is the unit of the semiring.

EXAMPLE 4.3 Let $\underline{\underline{\overline{N}}}$ denote the set of natural numbers $N = \{0,1,2,3,\ldots\}$ with an element $\infty \notin \underline{\underline{N}}$ appended. Because of notational conflicts, the semiring operators + and · will be denoted $\oplus$ and $\odot$ for this example only. Define

$$x \oplus y = \begin{cases} \min(x,y) \text{ if } x,y \in \underline{\underline{\overline{N}}} \\ x \text{ if } x \in \underline{\underline{N}} \text{ and } y = \infty \\ y \text{ if } y \in \underline{\underline{N}} \text{ and } x = \infty \\ \infty \text{ otherwise} . \end{cases} \qquad x \odot y = \begin{cases} x+y \text{ if } x,y \in N \\ \infty \text{ otherwise.} \end{cases}$$

It is easy to verify that $\underline{\underline{\overline{N}}}$ is a semiring. Note that $0 \in N$ is not the zero of the semiring, but rather ∞ is. Care must be taken to insure that there is no confusion. Also note that + is the multiplication in this example. This is why the symbols $\oplus$ and $\odot$ are used.

EXAMPLE 4.4 Let $\underline{\underline{B}} = \{0, 1\}$. On $\underline{\underline{B}}$ define

$$x + y = \begin{cases} 1 \text{ if } x = 1 \text{ or } y = 1 \\ 0 \text{ otherwise} , \end{cases} \qquad xy = \begin{cases} 1 \text{ if } x = 1 \text{ and } y = 1 \\ 0 \text{ otherwise.} \end{cases}$$

$\underline{\underline{B}}$ is a semiring, the usual binary logic.

Let X_1 and X_2 be monoids. A monoid morphism is a function $f:X_1 \to X_2$ such that $f(xy) = f(x)f(y)$ for all $x,y \in X$ and $f(1) = 1$. Let S_1 and S_2 be semirings, and let $f:S_1 \to S_2$. f is a semiring morphism if it is a monoid morphism for each of the two corresponding pairs of monoid structures, i.e.,

(a) $f(x+y) = f(x) + f(y)$ for all $x,y \in S_1$.

(b) $f(xy) = f(x)f(y)$ for all $x,y \in S_1$

(c) $f(0) = 0$; $f(1) = 1$.

f is an epimorphism (resp. monomorphism, resp. isomorphism) if it is surjective (resp. injective, resp. bijective).

EXAMPLE 4.5 Define $f:\underline{\underline{N}} \to \underline{\underline{B}}$ by

$$f(n) = \begin{cases} 1 \text{ if } n \neq 0 \\ 0 \text{ if } n = 0 \end{cases}$$

f is a semiring morphism.

EXAMPLE 4.6 Let S_1 and S_2 be semirings, and let $f:S_1 \quad S_2$ be a semiring morphism. Define $f[[x]]:S_1[[x]] \to S_2[(x)]$ by $\sum a_k x^k \rightsquigarrow \sum f(a_k) x^k$. $f[[x]]$ is a semiring morphism.

Free Modules Over Semirings

When doing system theory over fields, the concept of vector space over a field plays a central role. When dealing with system theory over semirings, the corresponding concept is that of module over a semiring. In this subsection, the basic ideas in modules over a semiring are presented.

Let S be a semiring. A module over S a commutative monoid $(M,+,0)$ (abbreviated M) a function $f: S \times M \rightarrow M$ such that for all $r,s \in S$ and $a,b \in M$:

(a) $f(r,a+b) = f(r,a) + f(r,b)$

(b) $f(r+s,a) = f(r,a) + f(s,a)$

(c) $f(rs,a) = f(r,f(s,a))$

(d) $f(1,a) = a$, $f(0,a) = 0$

Usually $f(r,m)$ is written as just rm.

EXAMPLE 4.7 Let S be a semiring, let $m \geq 0$ be an integer, and let S^m denote the m-fold cartesian product of S with itself. S^m is an S module under the operation $r(r_1,\ldots,r_m) = (rr_1,\ldots,rr_m)$. S^m is called the free S module of dimension m. Free S modules are the only kind of modules which will be considered in this report.

Let M_1 and M_2 be S modules. A function $f:M_1 \to M_2$ is an S module morphism if for all $r \in S$, $m, m' \in M_1$.

(a) $f(m+m') = f(m) + f(m')$

(b) $f(rm) = rf(m)$

For free modules the following result characterizes all S module morphisms.

THEOREM 4.8 Let S be a semiring, and let m,n be non-negative integers. There is a bijective correspondence between the set S(m,n) of all S module morphisms S^m S^n and the set $Mat_S(m,n)$ of all m×n matrices over S. The matrix A with entries a_{ij} is associated with the morphism $f_A:S^m \to S^n$ via matrix multiplication. That is, for $(s_1,\ldots,s_m) = s \in S^n$, $f_A(s)$ is given by

$$[s_1 \ldots s_m] \begin{bmatrix} a_{11} \ldots a_{1n} \\ \\ \\ a_{m1} \ldots a_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{ij} s_i \\ \\ \\ \sum_{i=1}^{m} a_{in} s_i \end{bmatrix}.$$

Proof: It is clear that each $F \in Mat_S(m,n)$ gives a unique $f \in S(m,n)$. Conversely, each morphism $f \in S(m,n)$ is uniquely determined by the elements $e_1 = f((1,0,\ldots,0))$, $e_2 = f((0,1,0,\ldots,0)),\ldots, e_m = f((0,0,\ldots,0,1))$, for then $f((s_1\ldots,s_m)) = \sum_{i=1}^{m} s_i f(e_i)$. Put $a_{ij} = (f(e_i))_j$.

Ordered Semirings

Frequently, semirings come with additional structure, and it is often possible to make effective use of this structure. One such additional structure which is particularly useful is a partial order. The ideas of partially-ordered semirings which are necessary for the results in interconnection theory to be developed here are given in this subsection. For a much more complete discussion of algebraic objects with order, consult [Birkhoff, 1967].

Let S be a semirings, and let $\leq$ be a partial order on S. $(S,\leq)$ is a partially ordered semiring if for all $s_1,s_2,s_3 \in S$, the following conditions are met:

(a) $s_2 \leq s_3 \Rightarrow s_1 + s_2 \leq s_1 + s_2$

(b) $s_2 \leq s_3 \Rightarrow s_1 s_2 \leq s_1 s_3$

If, in addition, $0 \in s$ for all $s \in S$, then $(S,\leq)$ is called positive. If every ascending chain $C = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq \ldots$ has a least upper bound in S, then $(S,\leq)$ is said to be σ-ordered. When no confusion can result, it will simply be said that S is a partially ordered (resp. positive, resp. σ-ordered) semiring, with the ordering $\leq$ understood. All of the partially-ordered semirings to be considered in this report are both positive and σ-ordered. These examples are now given.

EXAMPLE 4.9 On the semiring $\underline{\underline{B}}$ (example 4.4), let $0 \leq 1$. Then B is a positive σ-ordered semiring.

EXAMPLE 4.10 The semiring $\overline{\underline{\underline{N}}}$ of example 4.10, with the reverse ordering $\preccurlyeq$ given by

$$x \preccurlyeq y \Leftrightarrow x, y \in \underline{\underline{N}} \text{ and } y \leq x \text{ or } x = \infty .$$

Note that this is the reverse of the usual ordering, and care must be taken to avoid confusion. $\overline{\underline{\underline{N}}}$ is a positive σ-ordered semiring.

EXAMPLE 4.11 Let S be any partially-ordered semiring. Let S[[x]] be the ring of formal power series given in example 4.2. On S[[x]] define $\leq$ by $\sum a_k x^k \leq \sum b_k x^k$ if and only if $a_k \leq b_k$ for all k. Under this ordering, S[[x]] is a partially-ordered semiring. When S is positive (resp. σ-ordered), so is S[[x]], as all operations are pointwise.

It is not necessary here to develop the concept of partially-ordered semiring morphism, though such a definition is easy enough to develop. Rather, the concept of semimorphism is the one which turns out to be of importance, and will now be developed.

Let S_1 be any semiring, and let S_2 be a partially-ordered semiring. Let $f:S_1 \quad S_2$ be a function. f is called a <u>semiring semimorphism</u> if the following two conditions are satisfied for all $s,s' \in S_1$

(a) $f(s+s') \leq f(s) + f(s')$

(b) $f(ss') \leq f(s)f(s')$.

Notice that the concept of semimorphism is strictly weaker than that of morphism, i.e., all morphisms are semimorphisms. Nonetheless, a semimorphism preserves a certain amount of structure. Several important examples will now be given.

EXAMPLE 4.12 Let S be any semiring, and let $\underline{\underline{B}}$ be as in example 4.9. Define $f:S \rightarrow \underline{\underline{B}}$ by

$$f(s) = \begin{cases} 1 \text{ if } s \neq 0 \\ 0 \text{ if } s = 0 \end{cases} .$$

f is a semiring semimorphism. f is never a semiring morphism when $S \neq \{0\}$ is a ring, because for any $s \in S$, $s \neq 0$, $f(s+(-s)) = f(0) = 0 \neq 1 = f(s) + f(-s)$.

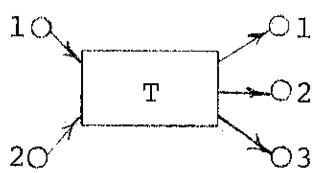
EXAMPLE 4.13 Let S be any semiring, and let S[[x]] be the semiring of example 4.2. Let $\underline{\underline{\bar{N}}}$ be as in example 4.10. Define $f:S[[x]] \rightarrow \underline{\underline{\bar{N}}}$ by $f(\sum a_k x^k) = \min(\{k \mid a_k \neq 0\})$. f is a semiring semimorphism.

EXAMPLE 4.14 Let S be any semiring, and let P be any partially-ordered semiring. Let $f:S \to P$ be a semiring semimorphism. The map $f[[x]]:S[[x]] \to P[[x]]$ of example 4.6 is a semiring semimorphism.

5. SIMPLIFICATION OF SYSTEMS DEFINED OVER SEMIRINGS

In this section, it will be shown that the interconnection properties of many large-scale dynamical systems may be accurately described by combining the mathematical theory of universal algebra with the mathematical theory of semirings.

Fix a semiring R. A m-input, n-output <u>system</u> <u>description</u> over R is an mxn matrix of elements from R. Such systems may be visualized as bipartite graphs. For example, let m-2, n=3. An m×n system description T is depicted as



The round nodes correspond to input and output terminals, while the square nodes correspond to the systems themselves. Suppose

$$t = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \end{bmatrix}$$

The element t_{ij} is called the <u>transfer</u> <u>element</u> from i to j, and represents the way a signal is transmitted from input i to output j. More precisely, inputs are taken from the space R^m

(in the example, R^2). The output resulting from input A is A T (matrix multiplication). For example if

$$A = [a_1, a_2]$$

then the output is

$$A\ T = [a_1\ a_2] \begin{bmatrix} t_{11} & t_{12} & t_{12} \\ T_{21} & t_{22} & t_{23} \end{bmatrix} = \begin{bmatrix} a_1 t_{11} + a_2 t_{21} \\ a_1 t_{12} + a_2 t_{22} \\ a_1 t_{13} + a_2 t_{23} \end{bmatrix}$$

EXAMPLE 5.1 Let S be any semiring (or, for the sake of concreteness, let R = $\underline{\underline{R}}$, the field of real numbers). Consider the ring S[[x]] of example 4.2. Regard each $\sum a_k x^k$ as a signal over discrete time. The k^{th} term is the value of the signal at time k. A (single-input, single-output) behavior is an S[[x]]-module morphism b:S[[x]]→S[[x]].

LEMMA 5.2 Let B:S[[x]]→S[[x]] be a function. B is a behavior (i.e., an S[[x]]-module morphism) if and only if the following two conditions are satisfied:

(a) B is an S-module morphism.

(b) B is a translation invariant, i.e., $b(x^* \sum a_k x^k) = x^* b(\sum a_k x^k)$.

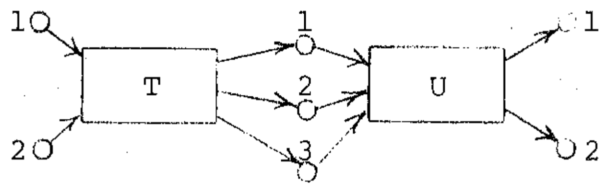
Proof: The proof is routine. See, e.g., [Eilenberg, 1974].

LEMMA 5.3 Let $B:S[[x]] \to S[[x]]$ be a behavior. Then B is causal, i.e., $a_k = b_k$ for $k < n \Rightarrow$ the terms for $k \leq n$ of $B(\sum a_k x^k)$ agree with those of $B(\sum b_k x^k)$.

Thus, behaviors $B:S[[x]] \to S[[x]]$ correspond intuitively to what one would consider behaviors of single-input, single-output systems to be. Now consider an m-input n-output system. A behavior of such a system is just an S[[x]] module morphism $B:(S[[x]])^m \to (S[[x]])^n$. By theorem 4.8, such a morphism is representable as an $m \times n$ matrix of elements from S[[x]]. Thus, by considering system descriptions over the semiring S[[x]], what are obtained are precisely the behaviors of multi-input, multi-output linear systems over S.

Basic Interconnections

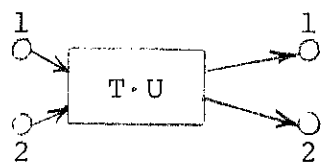
The utility of transfer elements arises in part from the fact that interconnection operators may be described naturally in terms of elementary matrix operations. Consider the operation of connecting two systems in series. Using bipartite graphs, this operation may be visualized as below, for the interconnection of a 2-input 3-output system T with a 3-input 2-output system U.



Notice that this operation only makes sense if the number of output lines of T is equal to the number of input lines of U. The transfer element of this interconnection is just the matrix product:

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{13} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} = \begin{bmatrix} t_{11}u_{11}+t_{12}u_{21}+t_{13}u_{31} & t_{11}u_{12}-t_{12}u_{32}+t_{13}u_{32} \\ t_{21}u_{11}+t_{22}u_{21}+t_{23}u_{31} & t_{21}u_{12}+t_{22}u_{22}+t_{23}u_{32} \end{bmatrix}$$

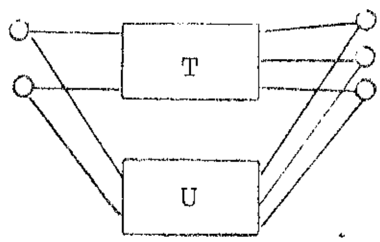
The above system is said to be <u>transfer equivalent</u> to T U, depicted



More generally, let S be any semiring. Recall that $Mat_s(m,n)$ is the set of all m×n matrices with entries from S. For each triple of natural numbers (m,n,p), there is a matrix multiplication map $ser(m,n,p):Mat_S(m,n)\times Mat_S(,np)\rightarrow Mat_S(m,p)$. This matrix multiplication operator is precisely the <u>series interconnection</u>

operator for system descriptions over S. It is easy to verify for the case of S[[x]] thst this operator corresponds exactly to interconnection of behaviors in the usual sense.

Next, consider the operation of connecting two systems in parallel. There are really two concepts of parallel which exist, and the framework developed here easily captures each of them. First, there is the operator of coupled parallel. In this interconnection, two system descriptions, matching in number of input lines as well as output lines, are interconnected. For two 2-input, 3-output systems T and U, the interconnection would be depicted as

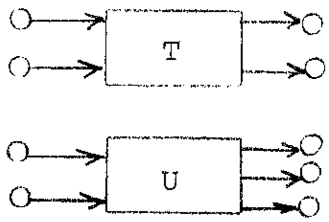


This corresponding operation is matrix addition:

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{12} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix} = \begin{bmatrix} t_{11}+u_{11} & t_{12}+u_{12} & t_{13}+u_{13} \\ t_{21}+u_{21} & t_{22}+u_{22} & t_{23}+u_{23} \end{bmatrix}$$

In the general case, coupled parallel of two m-input, n-output system descriptions is given by the matrix addition map $\mathrm{cpar}(m,n):\mathrm{Mat}_S(m,n) \times \mathrm{Mat}_S(m,n) \rightarrow \mathrm{Mat}_S(m,n)$. cpar(-,-) is called the coupled parallel interconnection operator.

The other way of connecting two systems in parallel is by using the <u>uncoupled parallel</u> operator. Any two system descriptions may be interconnected in this way. For a 2-input 2-output system connected in uncoupled parallel with a 2-input 3-output system, the bipartite graph description is



The corresponding operator is matrix cartesian product.

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{11} & t_{12} \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 & 0 \\ t_{21} & t_{22} & 0 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} & u_{13} \\ 0 & 0 & u_{21} & u_{22} & u_{23} \end{bmatrix}$$

In the general case, the coupled parallel of an m_1-input n_1-output system description with an m_2-input n_2-output system description is given by the matrix cartesian product operator $\mathrm{cart}(m_1,n_1,m_2,n_2)$: $\mathrm{Mats}(m_1,n_1) \times \mathrm{Mat}_S(m_2,n_2) \rightarrow \mathrm{Mat}_S(m_1+m_2,n_1+n_2)$.

<u>The Algebra of Interconnection Operators</u>

The operators which have just been described combine nicely to form a heterogeneous algebra. This algebra will now be described briefly. Let $M = \underline{\underline{N}} \times \underline{\underline{N}}$, i.e., the cartesian product of the set of natural numbers with itself. M will be the set of types. Define Ω on M by the following rules:

(a) for $m,n,p \in \underline{\underline{N}}$, $S(m,n,p) \in \Omega((m,n),(n,p)(m,p))$

(b) for $m,n \in \underline{\underline{N}}$, $C(m,n) \in \Omega((m,n)(m,n))$

(c) for $m,n,p,q \in \underline{\underline{N}}$, $U(m,n,p,q) \in \Omega((m,n),(p,q),(m+p,n+q))$

(d) Nothing else.

The following Ω equations E are also defined:

(e) $S(m,p,q)(S(m,n,p)(((m,n),v_1),((n,p),v_2))),((p,q),v_3)))$

$= S(m,n,q)(((m,n),v_1),S(n,p,q)(((n,p),v_2),((p,q),v_3)).$

(f) $C(m,n)(((m,n),v_1),((m,n),v_2)) = C(m,n)(((m,n),v_2)),((m,n)v_1)).$

(g) $C(m,n)(((m,n),v_1),C(m,n)((m,n),v_2),((m,n,v_3)))$

$= C(m,n)(C(((m,n),v_1)),((m,n),v_2)),((m,n)),v_3)).$

(h) $U(m,n,p+r,q+s)((((m,n),v_1),\ U(p,q,r,s)(((p,q),v_2),((r,s),v_3)))$

$= U(m+p,n+q,r,s)(U(m,n,p,q))(((m,n),v_1)),((p,q),v_2))),$

$((r,s),v_3)).$

The above equations are meant to hold for all $m,n,p,q,r,s, \in \underline{\underline{N}}$. S, C, and U are formal operators of series, coupled parallel, and uncoupled parallel, respectively. (e) says that series interconnection is associative; (f) says that coupled parallel interconnection is commutative and (g) says that it is associative; (h) says that uncoupled parallel interconnection is associative.

Now let S be any semiring. Let Mat_S denote the $M = \underline{\underline{N}} \times \underline{\underline{N}}$ set with $Mat_S((m,n)) = Mat_S(m,n)$, the set of all $m \times n$ matrices with elements in S. The (Ω, E) algebra (Mat_S, Int) is given by

$$Int_{S(m,n,p)} = ser(m,n,p)$$

$$Int_{C(m,n)} = cpar(m,n)$$

$$Int_{U(m,n,p,q)} = cart(m,n,p,q)$$

for all $m,n,p,q \in \underline{\underline{N}}$. This is just the algebra of series, coupled parallel, and uncoupled parallel interconnection of system descriptions in S.

Suppose now that S is a partially-ordered semiring. for each $(m,n) \in M$, an ordering is placed on $Mat_S(m,n)$ by $T \leq U$ if and only if $t_{ij} \leq u_{ij}$ for all (i,j), where t_{ij} and u_{ij} are , respectively, the ij entries of T and U. This ordering clearly makes Mat_S a partially-ordered M-set. Furthermore, the following is true.

LEMMA 5.4 Let S be a partially-ordered semiring. Then (Mat_S, Int) is a partially-ordered Ω algebra.

The proof is a simple consequence of the assumption that S is a partially-ordered semiring.

The topic to be considered next is that of algebraic interconnection spaces relative to a semiring S. Let $(C,(Mat_S,Int),g)$ be such a space (see section 2). Two simplifying assumptions will be made

(a) For each $(m,n) \in \underline{\underline{N}}\times\underline{\underline{N}}$, $C((m,n)) \in Mat_S(m,n)$.

(b) $g:C\rightarrow Mat_S$ is the inclusion (which will always be denoted inj).

Relabelling C is no problem; the only constraint here is that the original g be injective for each (m,n). This amounts to having only one label per component. These assumptions will be in force from now on.

Suppose that S and S' are semirings, and that $f:S\rightarrow S$ is a function. f induces an M function $Mat_f:Mat_S\rightarrow Mat_{S'}$ by pointwise substitution. That is, the matrix T whose (i,j) entry is $t_{ij} \in S$ is mapped to the matrix U whose (i,j) entry is $f(t_{ij}) \in S$. The correct notation for U is $Mat_f((m,n))(T)$ (assuming T is m×n), but this will often be abbreviated to f(T) when no confusion can result.

An important property of Mat_f is that it preserves algebraic operations. More precisely, the following is valid.

THEOREM 5.5 Let S and S' be semirings, and let $f:S \quad S'$ be a function. If f is a morphism of semirings, then Mat_f is an Ω-algebra morphism $Mat_f:(Mat_S,Int)\rightarrow(Mat_{S'},Int)$. Furthermore, if S' is a partially-ordered semiring and f is a semiring semimorphism, then Mat_f is an Ω-algebra semimorphism.

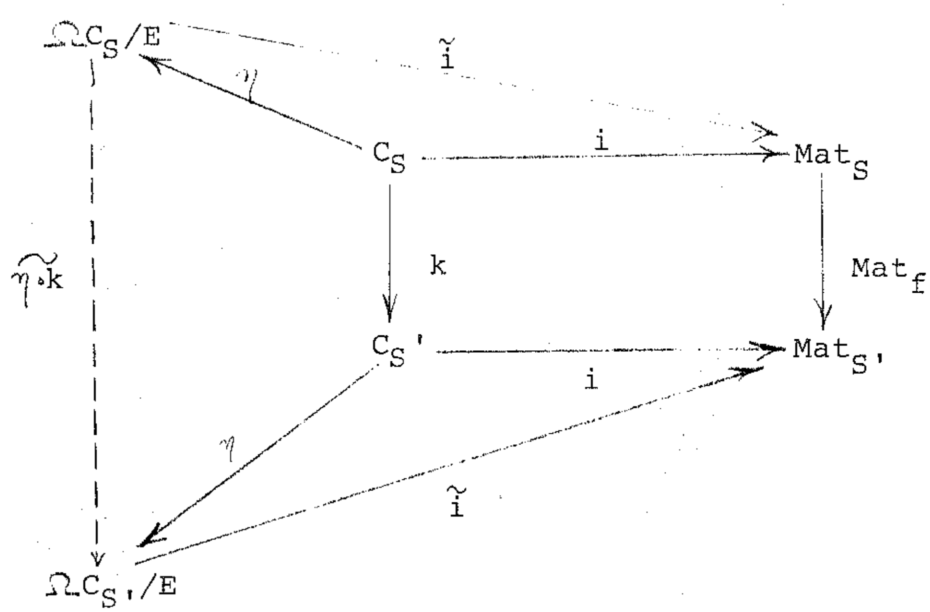
Proof: The proof the the above is completely straightforward verification.

COROLLARY 5.6 Let S be a semiring, let S' be a semiring (resp. partially-ordered semiring), let C_S and $C_{S'}$ be M-sets with $C_S((m,n)) \subset Mat_S(m,n)$ and $C_S((m,n)) \subset Mat_{S'}(m,n)$, and let $k:C_S \to C_{S'}$ be an M function. For k to be an exact interconnective modelling specifier (resp. semi-interconnective modelling specifier) from $(C_S,(Mat_S,Int),inj)$ to $(C_{S'},(Mat_{S'},Int),inj)$, it is sufficient that k be a restriction of Mat_f for some semiring morphism (resp. semimorphism) $f:S \to S'$.

Proof: If k is a restriction of Mat_f, then the diagram below commutes.

$$\begin{array}{ccc} C_S & \xrightarrow{\ i\ } & Mat_S \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle Mat_f} \\ C_{S'} & \xrightarrow{\ i\ } & Mat_{S'} \end{array}$$

Now using the universal properties of free algebras, the diagram below may be constructed.



From this diagram follows the result.

COROLLARY 5.7 In the situation of corollary 5.6 with f a semiring semimorphism, Mat_f is always optimal.

Proof: Note that the first diagram in 5.5 commutes, and consult theorem 3.1.

Examples of System Simplification

The algebra of interconnection operators just described is particularly interesting because of the interesting examples which fit into the framework. Some of these examples will now be discussed.

The systems to be simplified will generally be behaviors of discrete-time systems. That is, fix a semiring S. A total-structure behavior (relative to S) is an m n matrix of elements from S[[x]], interpreted as a behavior as illustrated earlier in this section. A total-structure interconnection space is completely given by an M subset $C \subset S[[x]]$. Think of C as M set of all "primitive" behaviors which will be interconnected. The corresponding interconnection space is $(C,(Mat_{S[[x]]},Int),inj)$.

A simplification of a total-structure interconnection space is given by a partially-ordered semiring S', a semiring semimorphism $f:S[[x]] \to S'$, and an M-function $k:C \to Mat_{S'}$ which is the restriction of Mat_f. By corollary 5.6, k is a semi-interconnective modelling specifier from $(C,(Mat_{S[[x]]},Int),inj)$ to $(Mat_{S'},(Mat_{S'},Int),inj)$. The intuition behind this modelling philosophy is very simple. S' is assumed to be a "simpler" semiring than S[[x]], yet it retains some of the properties of S[[x]].

EXAMPLE 5.8 Let S' be the semiring B (see example 4.4), and define $k:C \to B$ by $k(0)=0$ and $k(c)=1$ for $c \neq 0$. k is a semiring semimorphism (see example 4.12). This modeling strategy is called pure-structure modelling. The term is derived from the work of Siljak (see [Šiljak, 1977]), who first used this sort of binary labelling in the simplification of large-scale systems. The idea is that the system behavior matrix

$T \in Mat_{S[[x]]}$ is replaced by the "connection matrix" which contains ones where there is some interconnection and 0 if there is none. For example, working in $\underline{\underline{R}}[[x]]$, the 2-input 3-output system described by

$$\begin{bmatrix} 3x^2+2x & 4 \\ 9x^8 & 0 \end{bmatrix}$$

is simplified to

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The semimorphism property of Mat_f (where f is as in example 4.12) says that if system behaviors are interconnected, first computing the resulting behavior and then simplifying gives a smaller result than first substituting pure structure models for the components. The pure structure found by modelling with pure-structure components is thus a liberal model in the sense that it may show a connection where there is none. For example consider the problem of connecting in series the two behaviors shown below.

$$\begin{bmatrix} 3x & 18x^3+12x \\ 4x^2 & 3x^9 \end{bmatrix} \cdot \begin{bmatrix} 6x^2+4 & 6x^5 \\ 1 & -3/2x^5 \end{bmatrix} = \begin{bmatrix} 0 & 27x^8 \\ 3x^9+24x^4+16x^2 & -9/2\ x^{14}+24x^7 \end{bmatrix}$$

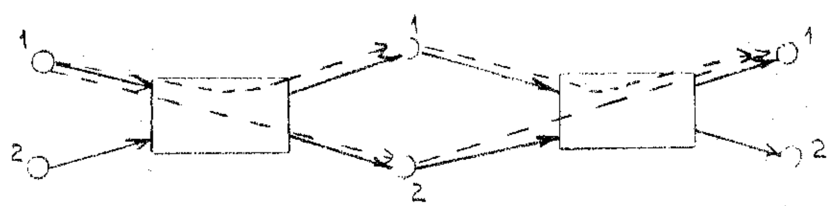
The pure structure of this interconnection is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

while first substituting pure structure and then interconnecting yields

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The simplification shows a path from input one to output one when there in fact is none. Viewing the problem in terms of directed bipartite graphs as illustrated below,



what has happened is that the dotted paths have cancelled each other out.

In a sense, the total structure modeling carries the maximal information of behavior of interconnections, while pure structure modeling contains the minimum useful information. The question naturally arises as to whether or not there is something in between. The answer is yes, and it will be illustrated by two examples.

EXAMPLE 5.9 Let S' be the semiring $\underline{\underline{B}}[[x]]$. Let $k:C \to \underline{\underline{B}}[[x]]$ be defined by $k(\sum a_k x^k) = \sum f(a_k) x^k$, where f is as in example 4.14. This example might well be called <u>local pure structure</u>. Instead of just telling whether or not there is a connection, it tells when there may be an interconnection, but nothing else. Using the matrix example of 5.8, the local pure structure of the interconnection is

$$\begin{bmatrix} 0 & x^8 \\ x^9+x^4+x^2 & x^{14}+x^7 \end{bmatrix}$$

while the local pure structure obtained by first substituting local pure structure and then interconnecting is

$$\begin{bmatrix} x & x^3+x \\ x & x^9 \end{bmatrix} \begin{bmatrix} x^2+x^0 & x^5 \\ x^0 & x^5 \end{bmatrix} = \begin{bmatrix} x^3+x & x^8+x^6 \\ x^9+x^4+x^2 & x^{14}+x^7 \end{bmatrix}$$

Again, it is seen that there is not equality, but only inequality.

EXAMPLE 5.10 Let S' be the semiring $\underline{\underline{N}}$. Let $k: C \to \underline{\underline{N}}$ be defined as in example 4.13. k takes $\sum a_k x^k \in S[[x]]$ to its first nonzero component. The interpretation is that it gives the minimal delay for an input to reach the appropriate output. It is called the <u>delay structure</u>. Again, it is not exact. Using once more the example from 5.8, the delay structure of the interconnection is

$$\begin{bmatrix} \infty & 8 \\ 2 & 7 \end{bmatrix},$$

while the interconnection of the component delay structures yields

$$\begin{bmatrix} 1 & 1 \\ 2 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 7 \end{bmatrix}$$

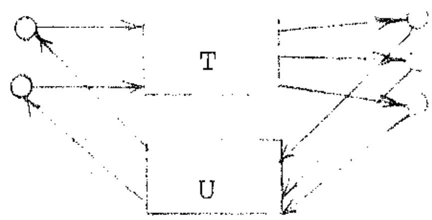
Thus, this is also only a semi-interconnective modeling situation, and not exact.

As a final remark, it is noted that each of the above modeling structures are optimal. This fact is a direct consequence of corollary 5.7.

6. SIMPLIFICATION OF SYSTEMS WITH FEEDBACK

In many cases, large-scale dynamical systems are connected not only in series and parallel as in the previous section, but in feedback as well. Feedback turns out to be substantially more difficult to handle than the previous two cases. Nonetheless, there are several techniques which do provide interesting results. In this section, one of the most useful of these techniques is described.

Fix a semiring S. The feedback interconnection which will be considered here is what will be called <u>totally-coupled feedback</u>. In this interconnection, an n-input, m-output system description T is connected to an n-input m-output system description U. For m=2 and n=3, such an interconnection would be depicted as below.



To write the corresponding matrix equation, assume

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{12} \\ t_{21} & t_{22} & t_{23} \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix}$$

are matrices with elements in S.

Tentatively, coupled feedback of T and U is any 2x3 matrix

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}$$

which satisfies the following equation:

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix} + \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{32} \end{bmatrix}$$

More generally, for any m×n matrix T and n×m matrix U, coupled feedback of T and U is tentatively defined to be any m×n matrix Y satisfying

$$Y = T + TUY \tag{1}$$

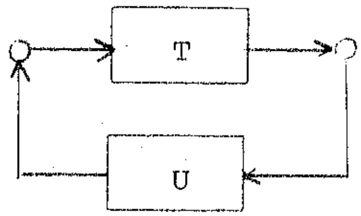
This definition is tentative because it raises two questions (a) might there be no solutions to the above equation, and (b) is it possible that there is more than one solution to the above equation? The answer is yes in each case. Thus the concept of coupled feedback must be defined more precisely. The basic approach here will be to attempt to give meaning to the series

$$T\left(\sum_{i=0}^{\infty} (UT)^{i}\right) .$$

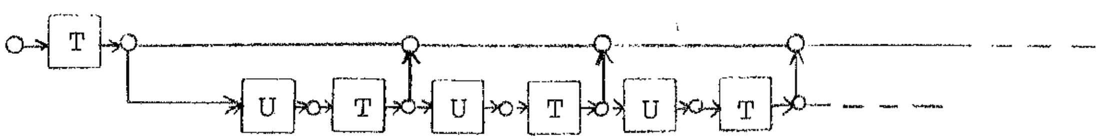
This seems reasonable because formal manipulation yields

$$T\left(\sum_{i=0}^{\infty} (UT)^i\right) = T + TU\left(T\sum_{i=0}^{\infty} (UT)^i\right)$$

so $T(\sum_{i=0}^{\infty} (TU)^i)$ is a reasonable candidate for a solution of (1). Of course, the formal series must be given precise meaning, and this will be done shortly. First, a physical interpretation of this series will be given. Consider a feedback connection of (for simplicity) two single-input. single output systems as shown below



A signal s entering is first transformed to T(s). It then exits, but also goes through U, then T,and then exits. It may repeat this looping path an arbitrary number of times. Using this analysis, an equivalent description of the above interconnection might be the infinite connection below.



However, this is just $T(\sum_{i=0}^{\infty} (UT)^i)$.

Sequential Covergence Structures

Let Y be any set, and let $Y^{\underline{\underline{N}}}$ denote the set of all sequences on Y. Let $\hat{Y} = Y \cup \{\perp\}$, where $\perp \notin Y$. A sequential convergence structure on Y is a function $c: Y^{\underline{\underline{N}}} \rightarrow Y$. Let $y \in Y^{\underline{\underline{N}}}$. If $c(y) \in Y$, it is said that y converges to c(y); if $c(y) = \perp$, it is said that y diverges. c(y) is often written as y_∞ when no confusion can result. The set $\{y \in Y^{\underline{\underline{N}}} \mid c(y) \in Y\}$ is called the convergence domain of c and is denoted $\mathcal{C}(c)$.

No suppose that S is a semiring, and let $y = y_o, y_1, y_2, \ldots \in S^{\underline{\underline{N}}}$. Note that S^N is isomorphic to S[[x]], via $y_o, y_1, y_2, \ldots \rightsquigarrow \sum y_k x^k$. Thus, $S^{\underline{\underline{N}}}$ is itself a semiring, and in particular an S-module under $a(y_o, y_1, y_2, \ldots) = (ay_o, ay_1, ay_2, \ldots)$. Let $C: S^{\underline{\underline{N}}} \rightarrow S$ be a sequential convergence structure. c is called morphic if the following two conditions are satisfied.

(a) $\mathcal{C}(c)$ is a subring of S.

(b) When restricted to $\mathcal{C}(c)$, c is an S-module morphism $S^{\underline{\underline{N}}} \rightarrow S$.

EXAMPLE 6.1 Let S be any semiring. Call $y = y_o, y_1, y_2, \ldots \in S^{\underline{\underline{N}}}$ almost constant if there is a $z \in S$ and $n \in \underline{\underline{N}}$ such that $y_n = z$ for all $m \geq n$. Define $c: S^{\underline{\underline{N}}} \rightarrow S$ by $c(y) = z$ (z as above) for y an almost-constant sequence, and $c(y) = \infty$ otherwise. Then c is a morphic sequential convergence structure on S.

EXAMPLE 6.2 Let S be any semiring, and consider S[[x]], Call $y = y_o, y_1, y_2, \ldots \in S[[x]]^{\underline{\underline{N}}}$ (where $y_i = \sum y_{ik}x^k$) <u>pointwise eventually constant</u> if for any $i \in N$ there is a $z \in S$ and $n \geq \underline{\underline{N}}$ such that $y_{im} = Z$ for $m \geq n$. Then c is a morphic sequential convergence structure on S[[x]].

In system modelling in this paper, the only semirings without a specified natural ordering which are used are those of the form S[[x]]. The above convergence structure will always be assumed for such semirings. However, on the positive σ-ordered semirings $\underline{\underline{B}}$, $\underline{\underline{N}}$, and $\underline{\underline{B}}[[x]]$, the natural ordering allows a much richer convergence structure to be assigned. This will now be detailed.

Let S be a positive σ-ordered semiring. Let $\uparrow(S)$ denote the set of all monotonically increasing sequences $a = a_o \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$ in S. For $a \in \uparrow(S)$, define a_∞ to be the least upper bound of a. Define $\sigma: S^{\underline{\underline{N}}} \to \hat{S}$ by $\sigma(a) = a_\infty$ if a is a montonic and $\sigma(a) =$ otherwise. σ is clearly a convergence structure on S, called the <u>natural σ-convergence structure</u> of S.

It does not seem to be necessarily true that σ is morphic. However, for the examples considered in this paper, the morphic property is not difficult to verify. Some general results about the morphic property are now presented.

LEMMA 6.3 Let S be a positive σ-ordered semiring, and let $\sigma: S^{\underline{\underline{N}}} \to S$ be the natural σ-convergence structure of S.

(a) $\mathcal{C}(\sigma)$ is a subring of $S^{\underline{\underline{N}}}$.

(b) For a pair $a,b \in \uparrow(s)$ $\sigma(a)+\sigma(b) \leq \sigma(a+b)$, $\sigma(a)\sigma(b) \leq \sigma(ab)$.

Proof: (a) If $a = a_o \leq a_1 \leq a_2 \leq \ldots$ and $b = b_o \leq b_1 \leq b_2 \leq \ldots$, clearly $a+b = a_o + b_o \leq a_1 + b_1 \leq a_2 + b_2 \leq \ldots$. For ab, note that

$$(ab)_n \quad = a_n b_o + a_{n-1} b_1 + \ldots + a_o b_n + \quad 0$$

$$(ab)_{n+1} = a_{n-1} b_o + a_n b_1 + \ldots + a_1 b_n + a_o b_{n+1}$$

Each term in the first line is less than the term in the second line which is under it, hence $(ab)_n \leq (ab)_{n+1}$. Thus ab is monotonic. (b) This is an immediate consequence of (a).

COROLLARY 6.4 Conditions as in 6.3, all that need be shown to guarantee that σ is morphic is that $\sigma(a+b) \leq \sigma(a) + \sigma(b)$ and $\sigma(a)\sigma(b) \leq \sigma(ab)$.

EXAMPLE 6.5 The natural σ-convergence structure on each of the semirings $\underline{\underline{B}}$, $\overline{\underline{\underline{N}}}$, and $\underline{\underline{B}}[[x]]$ is morphic.

Proof: The verification of 6.4 is trivial in these cases.

The next step is to extend the properties of a morphic sequential convergence structure on a semiring S to the algebra (Mat_S, Int) of section 5. This is really very easy; only the notation is cumbersome. The main idea will now be presented.

Let S be a semiring, and let $c: S^{\underline{\underline{N}}} \to S$ be a morphic sequential convergence structure on S. By doing everthing pointwise, this may be extended to a sequential convergence structure c(m,n) on $Mat_S(m,n)$ for any pair (m,n). Of course, $Mat_S(m,n)$ is not a semiring (unless m=n), so the definition of morphic is as it stands does not make any sense in this case. However, c is morphic in the following sense:

(a) $\mathcal{C}(c(m,n))$ is an S-submodule of S.

(b) When restricted to $\mathcal{C}(c(m,n))$, c(m,n) is an S-module morphism $Mat_S(m,n)^{\underline{\underline{N}}} \to Mat_S(m,n)$.

Using the above, it is easy to extend the idea of morphic sequential convergence structure to (Mat_S, Int). Each of the operators in (Mat_S, Int) is a binary operator of the form $h: Mat_S(m_1,n_1) \times Mat_S(m_2,n_2) \to Mat_S(m_3,n_3)$ for some $m_1, n_1, m_2, n_2, m_3, n_3 \in \underline{\underline{N}}$. h is <u>separately continuous</u> if for each $x = x_o, x_1, x_2, \ldots \in Mat_S(m_1,n_1)^{\underline{\underline{N}}}$ with x convergent to x_∞, and each $y \in Mat_S(m_2,n_2)$ the sequence $h(x_o,y)$, $h(x_1,)$, $h(x_2,y)$, ... converges to $h(x_\infty,y)$, and a similar condition holds for the second argument of h (keeping the first fixed).

Separate continuity means that the convergence property may be interchanged with the operations, as long as there is only one variable which is varying. This corresponds very closely with the usual concept of separate continuity of continuous functions between topological spaces. The following is immediate.

THEOREM 6.6 Let S be a semiring, and let $c: S^{\underline{\underline{N}}} \rightarrow S$ be a morphic convergence structure on S. With respect to the M set of extended convergence structures $c(m,n)$, each operator of (Mat_S, Int) is separately continuous.

Infinite Series in Sequential-Convergence Spaces

Using the concept of sequential convergence structure, it is very easy to define infinite series. Let S be a semiring, and let $c: S^{\underline{\underline{N}}} \rightarrow S$ be a morphic sequential convergence structure on S. Let $a_0, a_1, a_2, \ldots$ be a sequence in S. Define $s_n = \sum_{i=0}^{\infty} a_n$, and $s = s_0, s_1, s_2, \ldots$. The sum $\sum_{i=0}^{\infty} a_i$ is defined to be c(s) if it converges, and undefined otherwise. A similar definition is used for S module.

LEMMA 6.7 Let S be a positive σ-ordered semiring, and let $\sigma: S^{\underline{\underline{N}}}$ S be the natural σ-convergence structure. For any sequence $a_0, a_1, a_2, \ldots$ in S, $\sum_{i=0}^{\infty} a_i$ converges.

Proof: Clearly $\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i + 0 \leq \sum_{i=0}^{n} a_i + a_{n+1} = \sum_{i=0}^{n+1} a_i$, so $a_0, a_0+a_1, a_o+a_1+a_2 \ldots$ is monotonic. Hence it converges.

COROLLARY 6.8 Conditions as above, any series in $Mat_S(m,n)$ for any m,n converges.

In general, things are somewhat more difficult in general semirings with sequential convergence structures. Fortunately, for S[[x]], which is the central non-ordered semiring used in this paper, a very useful convergence theorem may be obtained.

Let S be a semiring. Call a sequence $a_o, a_1, a_2, \ldots$ in S[[x]] with $a_i = \sum a_{ik}x^k$ <u>eventually pointwise zero</u> if for each $n \in \underline{\underline{N}}$ ther is a $m \in n$ such that $k \geqslant m \Rightarrow a_{nk} = 0$.

THEOREM 6.9 Let S be a semiring, and let $a = a_0, a_1, a_2, \ldots$ be an eventually pointwise zero sequence in S[[x]]. Then $\sum_{i=0}^{\infty} a_i$ converges.

Proof: Define $s_n = \sum_{i=0}^{n} a_n$, and say $a_i = \sum a_{ik}x^k$, $s_{nk} = \sum_{i=0}^{k} a_{ni}$. Since a is eventually pointwise zero, $a_{ni} = 0$ for $i \geqslant m$ for some m. Hence s_n is pointwise eventually constant, and so converges.

Call a matrix $A \in Mat_S(m,n)$ pointwise eventually zero if each of its entries is such. The following is immediate.

COROLLARY 6.10 Let S be a semiring, and let $A_0, A_1, A_2, \ldots$ be a pointwise eventually zero sequence in $Mat_{S[[x]]}(m,n)$. Then $\sum_{i=0}^{\infty} A_i$ converges.

<u>Semi-Continuous Semimorphisms</u>

To complete the study of the mathematics required for feedback simplification of interconnected systems, it is necessary to be able to map limits of convergence structures in a nice way. A discussion of such mappings is now given.

Let S be a semiring and let c be a morphic sequential convergence structure on S. Let P be a positive σ-ordered semiring and let σ be the natural σ-convergence structure on P. Let $f:S \to P$ be a semiring semimorphism. f is <u>semicontinuous</u> if for any sequence $a = a_0, a_1, a_2, \ldots$ in S for which $\sum_{i=0}^{\infty} a_i$ converges, $\sum_{i=0}^{\infty} f(a_i)$ converges and $f(\sum_{i=0}^{\infty} a_i) \leq \sum_{i=0}^{\infty} f(a_i)$.

LEMMA 6.10 Let S be a semiring and let c be a morphic sequential convergence structure on S. Let P be a positive σ-ordered semiring and let σ be the natural σ-convergence structure on P. Let f be a semiring semimorphism which is semicontinuous. For any $m,n \in \underline{\underline{N}}$, if $A = A_1, A_1, A_2, \ldots$ is a sequence in $Mat_S(m,n)$ for which $\sum_{i=0}^{\infty} A_i$ converges, then $\sum_{i=0}^{\infty} f(A_i)$ converges in $Mat_P(m,n)$ and $f(\sum_{i=0}^{\infty} A_i) \leq \sum_{i=0}^{\infty} f(A_i)$.

Proof: The convergence in this case is just pointwise on the entries of the matrices, so the result follows immediately from the definition.

System Simplification

The mathematical machinery necessary to precisely solve the feedback interconnection problem posed at the beginning of this section has now been developed. Its use shall now be detailed.

Fix two semirings S and P, with P positive and σ-ordered. Assume that a morphic sequential convergence structure c is given for S, and that P has its natural σ-convergence structure. Also given is a semiring semimorphism $f:S \rightarrow P$ which is assumed to be semicontinuous. Suppose that $T \in Mat_S(m,n)$ and $U \in Mat_S(n,m)$ (with $m,n \in \underline{\underline{N}}$) are two system descriptions which are to be connected in coupled feedback in S. As a shorthand notation, elements will be just juxtaposed to denote an operator of the form ser(m,n,p), and + will be used to denote an operator of the form cpar(m,n). Also, X^n denotes XX...X (n times). Now define the <u>coupled</u> <u>feedback</u> of T and U to be the limit of the series $T \sum_{i=0}^{\infty} (UT)^i$ if it exists, and say that coupled feedback is undefined otherwise. Denote it cfb(m,n)(T,U). Since it is not always defined, this operator may not, strictly speaking, be added to the set of operators in (Mat_S, Int). However, it may be thought of as being so included, provided it is remembered that it is a partial operator.

NOTICE

"This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately-owned rights."

NOTICE

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161
Price: Printed Copy $; Microfiche $3.00

Page Range	Domestic Price	Page Range	Domestic Price
001–025	$ 4.00	326–350	$12.00
026–050	4.50	351–375	12.50
051–075	5.25	376–400	13.00
076–100	6.00	401–425	13.25
101–125	6.50	426–450	14.00
126–150	7.25	451–475	14.50
151–175	8.00	476–500	15.00
176–200	9.00	501–525	15.25
201–225	9.25	526–550	15.50
226–250	9.50	551–575	16.25
251–275	10.75	576–600	16.50
276–300	11.00	601–up	[1]
301–325	11.75		

[1] Add $2.50 for each additional 100 page increment from 601 pages up.

Corrections to:

"Algebraic simplification of interconnected systems"
by Stephen J. Hegner

These corrections also apply to Chapter 3 of the composite report "Towards a mathematical theory of modeling" by G. C. Corynen, S. Aggarwal, and S. J. Hegner.

The location of each error is listed for each of the three cases:

original report;
full-sized composite report;
reduced (double column) composite report;

in that order.

Key: p. = page; c. = column; l. = line; lb. = line back (count up from bottom of page).

p. 2-2, l. 2: Change "$\underline{\underline{N}}$ $\underline{\underline{N}}$" to "$\underline{\underline{N}} \times \underline{\underline{N}}$".
p. 49, l. 2:
p. 12, c. 2, l. 26:

p. 2-2, lb. 5: Change "M-operator domain" to "M-<u>operator</u> <u>domain</u>".
p. 49, lb. 5:
p. 12, c. 2, lb. 5:

p. 2-3, l. 6: Change "Ω((m,n)(m,p),(m,p))" to "$\underline{\Omega}$((m,n),(n,p),(m,p))".
p. 50, l. 6:
p. 13, c. 1, l. 6:

p. 2-4, lb. 2: Change "h" to "$\tilde{h}$".
p. 51, lb. 2:
p. 13, c. 2, lb. 8:

p. 2-10, lb. 5: Change "h" to "$\tilde{h}$".
p. 57. lb. 5:
p. 14, c. 2, lb. 5:

p. 3-2, first diagram: Change " $_C{}^{k=\bar{k}}$" to "$\eta_C{}^{\cdot k=\bar{k}}$".
p. 59, first diagram:
p.14, c. 2, diagram at bottom:

p. 3-2, l. 5: Change "g; $(Y_1, {}_1)$ $(Y_2, {}_2)$" to
p. 59, l. 5:
p. 15, c. 1, l. 5: "g: $(Y_1,\gamma_1) \rightarrow (Y_2,\gamma_2)$".

p. 3-6, l. 7: Insert "$\rightarrow$" between "(Y_1,γ_1)" and "(Y_2,γ_2)".
p. 63, l. 7:
p. 15, c. 2, l. 9:

p. 4-2, l. 4: Change "x X" to "$x \in X$".
p. 66, l. 4:
p. 16, c. 1, lb. 10:

p. 4-3, l. 3: Change ":" to "."; "+" to "*".
p. 67, l. 3:
P. 16. c. 2, l. 13:

p. 4-4, 1.3: Insert "$\rightarrow$" between "S_1" and "S_2".
p. 68, 1. 3:
P. 17, c. 1, 1. 2:

p. 4-6, 1. 9: Insert "$\rightarrow$" between "S^m" and "S^n".
p. 70, 1. 9:
p. 17, c. 1, 1b. 9:

p. 5-1, 1. 8: Change "m n" to "m×n".
p. 75, 1. 8:
p. 18, c.1, 1b. 13:

p. 5-2, 1. 5: Change "A T" to "A·T".
p. 76, 1. 5:
p. 18, c. 2, 1. 2:

p. 5-4, 1b. 5: Change "$Mat_s(m,n)$" to "$Mat_S(m,n)$".
p. 78, 1b. 5:
p. 19, c. 1, 1. 2:

p. 5-4, 1b. 2: Change "$Mat_S(,np)$" to "$Mat_S(n,p)$".
p. 78, 1b. 2:
p. 19, c. 1, 1. 5:

p. 5-8, 1.11: Change "Mat_s" to "Mat_S".
p. 82, 1. 1:
p. 19, c. 2, 1. 3:

p. 5-8, 1b. 3: Change "Mat_s" to "Mat_S".
p. 82, 1b. 3:
p. 19, c. 2, 1b. 3:

p. 5-9, 1. 2: Change "Mat_s" to "Mat_S".
p. 83, 1. 2:
p. 20, c. 1, 1. 2:

p. 5-12, 1. 3: Change "m n" to "m×n".
p. 86, 1. 3:
p. 20, c. 2, 1. 13:

p. 6-4, 1. 10: Change "S^N" to "$S^{\underline{N}}$".
p. 94, 1. 10:
p. 22, c. 2, 1. 5:

p. 6-5, 1b. 7: Change "is a monotonic" to "is monotonic".
p. 95, 1b. 7:
p.22, c. 2, 1b. 15:

p. 6-5, 1b. 6: Insert "$\perp$" between "$\neq$" and "otherwise".
p. 95, 1b. 6:
p. 22, c. 2, 1b. 14: