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On the sensitivity of the spectral projection ☆

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Abstract

In this paper, we apply the theory of condition developed by Rice to define condition numbers of the spectral projection. Explicit expressions of the condition numbers are derived, and some relations between the condition numbers of the spectral projection and the condition number of the associated invariant subspace are presented. The results are illustrated by a simple numerical example.

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1. Introduction

Throughout this paper, $\mathscr{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices. A^{T} denotes the transpose of a matrix A, $\overline{A}$ the conjugate of A, and $A^H = \overline{A}^{\mathrm{T}}$. $A^\dagger$ stands for the Moore–Penrose inverse of A. I_n is the identity matrix of order n, and 0 is the null matrix. $\mathscr{R}(A)$ is the column space of A. $\lambda(A)$ denotes the set of all eigenvalues of a square matrix A. $\sigma_{\min}(A)$ denotes the smallest singular value of A. The

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symbol $\| \|_2$ stands for the Euclidean vector norm and the spectral matrix norm, and $\| \|_{\mathrm{F}}$ the Frobenius norm. For $A = (\alpha_{jk}) = (a_1, \ldots, a_n) \in \mathscr{C}^{m\times n}$ and a matrix B, $A \otimes B = (\alpha_{jk} B)$ is a Kronecker product, and vec A is a vector defined by vec $A = (a_1^{\mathrm{T}}, \ldots, a_n^{\mathrm{T}})^{\mathrm{T}}$.

Let $A \in \mathscr{C}^{n\times n}$, and let $U \in \mathscr{C}^{n\times n}$ be a unitary matrix such that

$$A = U \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} U^H, \tag{1.1}$$

where $A_{11} \in \mathscr{C}^{m\times m}$ $(m < n)$ [4]. Assume $\lambda(A_{11}) \bigcap \lambda(A_{22}) = \emptyset$. Then the Sylvester equation

$$A_{11}M - MA_{22} = -A_{12}, \tag{1.2}$$

has a unique solution M. Taking the solution M, and letting

$$\begin{aligned} S &= U \begin{pmatrix} I_m & M \\ 0 & I_{n-m} \end{pmatrix} = (S_1, S_2), \\ T &= S^{-1} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \quad S_1, T_1^{\mathrm{T}} \in \mathscr{C}^{n\times m}, \end{aligned} \tag{1.3}$$

we have

$$A = S \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} S^{-1}, \tag{1.4}$$

which means that $\mathscr{R}(S_1)$ is the invariant subspace of A corresponding to $\lambda(A_{11})$. The spectral projection of A corresponding to $\lambda(A_{11})$ is defined by

$$P = S \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = U \begin{pmatrix} I_m & -M \\ 0 & 0 \end{pmatrix} U^H; \tag{1.5}$$

i.e., the spectral projection P is the projection onto $\mathscr{R}(S_1)$ along $\mathscr{R}(S_2)$. It is known [1,3,5,10] that the spectral projection P plays an important role in the perturbation theory for eigenvalue problems.

Take a closed Jordan curve γ that separates the sets $\lambda(A_{11})$ and $\lambda(A_{22})$ in the complex plane, and let the domain containing $\lambda(A_{11})$ be to the left if we move in the counterclockwise direction. Then the spectral projection P defined by (1.5) can be expressed by a complex contour integral along γ [2,5]:

$$P = \frac{1}{2\pi i} \oint_\gamma (zI_n - A)^{-1} \mathrm{d}z. \tag{1.6}$$

The integral representation (1.6) has been used to develop numerical methods for computing the spectral projection P and to derive some perturbation bounds for P (see, e.g., [3, Section 2] and [2, Section 8.3]).

The purpose of this paper is to define condition numbers of the spectral projection P, and to derive explicit expressions of the condition numbers.

In Section 2, we apply the theory of condition developed by Rice [6] to define absolute and relative condition numbers of the spectral projection P, and derive explicit expressions of the condition numbers by using the matrix representation (1.5). Some relations between the condition numbers of the spectral projection P and the condition number of the associated invariant subspace $\mathscr{R}(S_1)$ are presented (see Remark 2.4 of Section 2). The results are illustrated by a simple numerical example in Section 3.

The following result cited from Stewart [7] will be used in Sections 2.

Theorem 1.1 [7, Theorem 4.11]. *Let*

$$\begin{aligned} &A_{11}, \Delta A_{11} \in \mathscr{C}^{m\times m}, \quad A_{22}, \Delta A_{22} \in \mathscr{C}^{(n-m)\times(n-m)}, \\ &\Delta A_{12} \in \mathscr{C}^{m\times(n-m)}, \quad \Delta A_{21} \in \mathscr{C}^{(n-m)\times m}. \end{aligned} \tag{1.7}$$

Assume $\lambda(A_{11}) \bigcap \lambda(A_{22}) = \emptyset$, *and define*

$$\delta = \sigma_{\min}(I_{n-m} \otimes A_{11} - A_{22}^{\mathrm{T}} \otimes I_m) - (\|\Delta A_{11}\|_{\mathrm{F}} + \|\Delta A_{22}\|_{\mathrm{F}}). \tag{1.8}$$

If

$$\delta > 0 \quad and \quad \frac{\|\Delta A_{12}\|_{\mathrm{F}}\|\Delta A_{21}\|_{\mathrm{F}}}{\delta^2} < \frac{1}{4},$$

then there is a unique solution $X \in \mathscr{C}^{(n-m)\times m}$ *to the equation*

$$A_{11}X - XA_{22} = -\Delta A_{12} + X\Delta A_{22} - \Delta A_{11}X + X\Delta A_{21}X$$

that satisfies

$$\|X\|_{\mathrm{F}} < \frac{2\|\Delta A_{12}\|_{\mathrm{F}}}{\delta}.$$

Let A_{11} and A_{22} be the matrices of (1.7). By Stewart [7], the separation $\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22})$ of the matrices A_{11} and A_{22} is defined by

$$\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}) = \min_{\|G\|_{\mathrm{F}}=1} \|A_{11}G - GA_{22}\|_{\mathrm{F}}. \tag{1.9}$$

Let

$$\Gamma = I_{n-m} \otimes A_{11} - A_{22}^{\mathrm{T}} \otimes I_m, \tag{1.10}$$

and assume $\lambda(A_{11}) \bigcap \lambda(A_{22}) = \emptyset$. Then from (1.9) we see that $\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22})$ has the expression

$$\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}) = \|\Gamma^{-1}\|_2^{-1} = \sigma_{\min}(\Gamma). \tag{1.11}$$

Thus, (1.8) can be written as

$$\delta = \mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}) - (\|\Delta A_{11}\|_{\mathrm{F}} + \|\Delta A_{22}\|_{\mathrm{F}}).$$

2. Condition numbers of the spectral projection *P*

Let the matrix A be slightly perturbed to $\tilde{A} = A + \Delta A$, and let the spectral projection P be perturbed to $\tilde{P} = P + \Delta P$, correspondingly. By the theory of condition developed by Rice [6] we may define the absolute and relative condition numbers $c_{\mathrm{abs}}(P)$ and $c_{\mathrm{rel}}(P)$ by

$$c_{\mathrm{abs}}(P) = \lim_{\delta\to 0} \sup_{\|\Delta A\|_{\mathrm{F}}\leqslant\delta} \frac{\|\Delta P\|_{\mathrm{F}}}{\delta}, \tag{2.1}$$

and

$$c_{\mathrm{rel}}(P) = \lim_{\delta\to 0} \sup_{\frac{\|\Delta A\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}}\leqslant\delta} \frac{\|\Delta P\|_{\mathrm{F}}}{\|P\|_{\mathrm{F}}\delta}. \tag{2.2}$$

By the definitions (2.1) and (2.2), we have the first order perturbation bounds for P:

$$\|\tilde{P} - P\|_{\mathrm{F}} \lesssim c_{\mathrm{abs}}(P)\|\Delta A\|_{\mathrm{F}}, \qquad \frac{\|\tilde{P} - P\|_{\mathrm{F}}}{\|P\|_{\mathrm{F}}} \lesssim c_{\mathrm{rel}}(P)\frac{\|\Delta A\|_{\mathrm{F}}}{\|A\|_{\mathrm{F}}},$$

where $\|\Delta A\|_{\mathrm{F}}$ is sufficiently small.

In this section, we will derive explicit expressions of $c_{\mathrm{abs}}(P)$ and $c_{\mathrm{rel}}(P)$.

Write

$$S^{-1}\Delta A S = \begin{pmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{pmatrix}, \quad \Delta A_{11} \in \mathscr{C}^{m\times m}. \tag{2.3}$$

Combining (2.3) with (1.4) gives

$$S^{-1}(A + \Delta A)S = \begin{pmatrix} A_{11} + \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & A_{22} + \Delta A_{22} \end{pmatrix}. \tag{2.4}$$

Consider the equations

$$A_{11}X - XA_{22} = -\Delta A_{12} + X\Delta A_{22} - \Delta A_{11}X + X\Delta A_{21}X, \tag{2.5}$$

and

$$A_{22}Z - ZA_{11} = -\Delta A_{21} + Z\Delta A_{11} - \Delta A_{22}Z + Z\Delta A_{12}Z. \tag{2.6}$$

By Theorem 1.1, if $\|\Delta A\|_{\mathrm{F}}$ is sufficiently small, then there is a unique solution $X \in \mathscr{C}^{m\times(n-m)}$ to Eq. (2.5) that satisfies

$$\|X\|_{\mathrm{F}} = \mathrm{O}(\|\Delta A\|_{\mathrm{F}}), \quad \text{as } \|\Delta A\|_{\mathrm{F}} \to 0, \tag{2.7}$$

and there is a unique solution $Z \in \mathscr{C}^{(n-m)\times m}$ to Eq. (2.6) that satisfies

$$\|Z\|_{\mathrm{F}} = \mathrm{O}(\|\Delta A\|_{\mathrm{F}}), \quad \text{as } \|\Delta A\|_{\mathrm{F}} \to 0. \tag{2.8}$$

Let $\|\Delta A\|_{\mathrm{F}}$ be so small that the unique solutions X and Z satisfy $\|X\|_2 < 1$ and $\|Z\|_2 < 1$. Then

$$\begin{pmatrix} I_m & X \\ Z & I_{n-m} \end{pmatrix} \text{ is nonsingular,}$$

and we have

$$\begin{pmatrix} I_m & X \\ Z & I_{n-m} \end{pmatrix}^{-1} = \begin{pmatrix} (I_m - XZ)^{-1} & 0 \\ 0 & (I_{n-m} - ZX)^{-1} \end{pmatrix} \begin{pmatrix} I_m & -X \\ -Z & I_{n-m} \end{pmatrix},$$

and

$$\begin{aligned} &\begin{pmatrix} I_m & X \\ Z & I_{n-m} \end{pmatrix}^{-1} \begin{pmatrix} A_{11} + \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & A_{22} + \Delta A_{22} \end{pmatrix} \begin{pmatrix} I_m & X \\ Z & I_{n-m} \end{pmatrix} \\ &\quad = \begin{pmatrix} (I_m - XZ)^{-1}(A_{11} + \widehat{\Delta A_{11}}) & 0 \\ 0 & (I_{n-m} - ZX)^{-1}(A_{22} + \widehat{\Delta A_{22}}) \end{pmatrix}, \end{aligned} \tag{2.9}$$

where

$$\widehat{\Delta A_{11}} = \Delta A_{11} - X \Delta A_{21} + \Delta A_{12} Z - X(A_{22} + \Delta A_{22}) Z,$$

$$\widehat{\Delta A_{22}} = \Delta A_{22} - Z \Delta A_{12} + \Delta A_{21} X - Z(A_{11} + \Delta A_{11}) X.$$

Moreover, let $\|\Delta A\|_{\mathrm{F}}$ be so small that

$$\lambda\left((I_m - XZ)^{-1}(A_{11} + \widehat{\Delta A_{11}})\right) \bigcap \lambda\left((I_{n-m} - ZX)^{-1}(A_{22} + \widehat{\Delta A_{22}})\right) = \emptyset, \tag{2.10}$$

and let

$$\tilde{S} = S \begin{pmatrix} I_m & X \\ Z & I_{n-m} \end{pmatrix}. \tag{2.11}$$

Then from (2.4) and (2.9) we get

$$\begin{aligned} &\tilde{S}^{-1}(A + \Delta A)\tilde{S} \\ &\quad = \begin{pmatrix} (I_m - XZ)^{-1}(A_{11} + \widehat{\Delta A_{11}}) & 0 \\ 0 & (I_{n-m} - ZX)^{-1}(A_{22} + \widehat{\Delta A_{22}}) \end{pmatrix}. \end{aligned} \tag{2.12}$$

Consequently, from (2.10)–(2.12) we see that the perturbed spectral projection $\tilde{P}$ can be expressed by

$$\begin{aligned} \tilde{P} &= \tilde{S} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \tilde{S}^{-1} \\ &= S \begin{pmatrix} (I_m - XZ)^{-1} & -(I_m - XZ)^{-1} X \\ Z(I_m - XZ)^{-1} & -Z(I_m - XZ)^{-1} X \end{pmatrix} S^{-1}. \end{aligned} \tag{2.13}$$

Combining (2.13) with (1.5) gives

$$\Delta P = S \begin{pmatrix} XZ(I_m - XZ)^{-1} & -X - XZ(I_m - XZ)^{-1}X \\ Z + ZXZ(I_m - XZ)^{-1} & -Z(I_m - XZ)^{-1}X \end{pmatrix} S^{-1}. \tag{2.14}$$

By (2.5) and (2.7), the vector vec X has the first order perturbation expansion

$$\operatorname{vec} X \approx -\Gamma^{-1}\operatorname{vec}(\Delta A_{12}), \tag{2.15}$$

where

$$\Gamma = I_{n-m} \otimes A_{11} - A_{22}^{\mathrm{T}} \otimes I_m. \tag{2.16}$$

By (2.6) and (2.8), the vector vec Z has the first order perturbation expansion

$$\operatorname{vec} Z \approx \Omega^{-1}\operatorname{vec}(\Delta A_{21}), \tag{2.17}$$

where

$$\Omega = A_{11}^{\mathrm{T}} \otimes I_{n-m} - I_m \otimes A_{22}. \tag{2.18}$$

Substituting (1.3), (2.7), (2.8), (2.15) and (2.17) into (2.14) gives the first order perturbation expansion of vec(ΔP):

$$\begin{aligned} \operatorname{vec}(\Delta P) &\approx \operatorname{vec}(-S_1XT_2 + S_2ZT_1) \\ &= -(T_2^{\mathrm{T}} \otimes S_1)\operatorname{vec} X + (T_1^{\mathrm{T}} \otimes S_2)\operatorname{vec} Z \\ &\approx (T_2^{\mathrm{T}} \otimes S_1)\Gamma^{-1}\operatorname{vec}(\Delta A_{12}) + (T_1^{\mathrm{T}} \otimes S_2)\Omega^{-1}\operatorname{vec}(\Delta A_{21}). \end{aligned} \tag{2.19}$$

By (1.3) and (2.3) we have

$$\operatorname{vec}(\Delta A_{12}) = (S_2^{\mathrm{T}} \otimes T_1)\operatorname{vec}(\Delta A), \quad \operatorname{vec}(\Delta A_{21}) = (S_1^{\mathrm{T}} \otimes T_2)\operatorname{vec}(\Delta A).$$

Combining these relations with (2.19) shows

$$\operatorname{vec}(\Delta P) \approx \Phi\operatorname{vec}(\Delta A), \tag{2.20}$$

where

$$\Phi = (T_2^{\mathrm{T}} \otimes S_1)\Gamma^{-1}(S_2^{\mathrm{T}} \otimes T_1) + (T_1^{\mathrm{T}} \otimes S_2)\Omega^{-1}(S_1^{\mathrm{T}} \otimes T_2). \tag{2.21}$$

Further, substituting the expression (2.20) into (2.1) and (2.2) gives

$$c_{\mathrm{abs}}(P) = \sup_{\|\Delta A\|_{\mathrm{F}} \leqslant 1} \frac{\|\Phi\operatorname{vec}(\Delta A)\|_2}{\|\Delta A\|_{\mathrm{F}}} = \|\Phi\|_2, \tag{2.22}$$

and

$$c_{\mathrm{rel}}(P) = \frac{\|A\|_{\mathrm{F}}\|\Phi\|_2}{\|P\|_{\mathrm{F}}}. \tag{2.23}$$

Observe that by (1.3) we have the expressions

$$S_1 = U\begin{pmatrix} I_m \\ 0 \end{pmatrix}, \quad S_2 = U\begin{pmatrix} M \\ I_{n-m} \end{pmatrix},$$

and

$$T_1 = (I_m, -M)U^H, \quad T_2 = (0, I_{n-m})U^H.$$

Substituting these expressions into (2.21) yields

$$\begin{aligned}\Phi = (\overline{U} \otimes U) \Bigg\{ & \left[\begin{pmatrix} 0 \\ I_{n-m} \end{pmatrix} \otimes \begin{pmatrix} I_m \\ 0 \end{pmatrix} \right] \Gamma^{-1} \left[(M^{\mathrm{T}}, I_{n-m}) \otimes (I_m, -M) \right] \\ & + \left[\begin{pmatrix} I_m \\ -M \end{pmatrix} \otimes \begin{pmatrix} M \\ I_{n-m} \end{pmatrix} \right] \Omega^{-1} \left[(I_m, 0) \otimes (0, I_{n-m}) \right] \Bigg\} (U^{\mathrm{T}} \otimes U^H),\end{aligned}$$

where $\overline{U} \otimes U$ and $U^{\mathrm{T}} \otimes U^H$ are unitary matrices. Hence, from (2.21) to (2.23) we obtain

$$c_{\mathrm{abs}}(P) = \left\| \Phi^{(0)} \right\|_2, \quad c_{\mathrm{rel}}(P) = \frac{\|A\|_{\mathrm{F}} \left\| \Phi^{(0)} \right\|_2}{\|P\|_{\mathrm{F}}}, \tag{2.24}$$

where

$$\begin{aligned}\Phi^{(0)} = & \left[\begin{pmatrix} 0 \\ I_{n-m} \end{pmatrix} \otimes \begin{pmatrix} I_m \\ 0 \end{pmatrix} \right] \Gamma^{-1} [(M^{\mathrm{T}}, I_{n-m}) \otimes (I_m, -M)] \\ & + \left[\begin{pmatrix} I_m \\ -M^{\mathrm{T}} \end{pmatrix} \otimes \begin{pmatrix} M \\ I_{n-m} \end{pmatrix} \right] \Omega^{-1} [(I_m, 0) \otimes (0, I_{n-m})].\end{aligned} \tag{2.25}$$

Overall, we have proved the following result.

Theorem 2.1. *The condition numbers* $c_{\mathrm{abs}}(P)$ *and* $c_{\mathrm{rel}}(P)$ *defined by* (2.1) *and* (2.2) *have the explicit expressions* (2.24), *where* $\Phi^{(0)}$ *is the matrix defined by* (2.25), *in which* M *is the unique solution to Eq.* (1.2), *and* Γ *and* Ω *are the matrices defined by* (2.16) *and* (2.18), *respectively.*

Remark 2.2. Consider a very simple example. Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{with } \alpha > 0.$$

By (1.1), (1.2) and (1.5), we have $U = I_2$ and $M = 0$, and the spectral projection P of A corresponding to the eigenvalue $\lambda = 0$ is

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, by (2.16), (2.18) and (2.25), we have

$$\Gamma = \Omega = -\alpha, \quad \Phi^{(0)} = -\frac{1}{\alpha} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, by Theorem 2.1, the condition numbers $c_{\text{abs}}(P)$ and $c_{\text{rel}}(P)$ can be expressed by

$$c_{\text{abs}}(P) = \frac{1}{\alpha} \quad \text{and} \quad c_{\text{rel}}(P) = 1, \tag{2.26}$$

which mean that the spectral projection P of this example is ill-conditioned in the absolute sense when α is very small, and P is always well-conditioned for $\alpha > 0$ in the relative sense. Note that the condition numbers of (2.26) are attainable. In fact, if the matrix A is perturbed to

$$\tilde{A} = \begin{pmatrix} 0 & \epsilon \\ 0 & \alpha \end{pmatrix},$$

then the spectral projection P is perturbed to

$$\tilde{P} = \begin{pmatrix} 1 & -\epsilon/\alpha \\ 0 & 0 \end{pmatrix},$$

and we have

$$\frac{\|\tilde{P} - P\|_{\text{F}}}{\|\tilde{A} - A\|_{\text{F}}} = \frac{|\epsilon|/\alpha}{|\epsilon|} = \frac{1}{\alpha}.$$

Remark 2.3. We now give some estimates of the condition number $c_{\text{abs}}(P)$.

Let

$$K_0 = \begin{pmatrix} I_m \\ -M^{\text{T}} \end{pmatrix} \otimes \begin{pmatrix} M \\ I_{n-m} \end{pmatrix}, \quad L_0 = (M^{\text{T}}, I_{n-m}) \otimes (I_m, -M), \tag{2.27}$$

$$K = \left(\begin{pmatrix} 0 \\ I_{n-m} \end{pmatrix} \otimes \begin{pmatrix} I_m \\ 0 \end{pmatrix}, K_0 \right) \in \mathscr{C}^{n^2 \times 2m(n-m)}, \tag{2.28}$$

and

$$L = \begin{pmatrix} L_0 \\ (I_m, 0) \otimes (0, I_{n-m}) \end{pmatrix} \in \mathscr{C}^{2m(n-m) \times n^2}. \tag{2.29}$$

Note that the matrix K has full column rank. In fact, the relation $Kx = 0$ for $x = (x_1^{\text{T}}, x_2^{\text{T}})^{\text{T}}$ with $x_j \in \mathscr{C}^{m(n-m)} (j = 1, 2)$ can be written as

$$\begin{pmatrix} I_m \\ 0 \end{pmatrix} X_1 (0, I_{n-m}) + \begin{pmatrix} M \\ I_{n-m} \end{pmatrix} X_2 (I_m, -M) = 0, \quad \operatorname{vec} X_j = x_j, \ j = 1, 2,$$

i.e.,

$$\begin{pmatrix} M X_2 & X_1 - M X_2 M \\ X_2 & -X_2 M \end{pmatrix} = 0,$$

which implies $X_2 = 0$ and $X_1 = 0$. Consequently, K has full column rank. Similarly, we can prove that L has full row rank.

By using (2.28) and (2.29), the matrix $\Phi^{(0)}$ defined by (2.25) can be expressed by

$$\Phi^{(0)} = K \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix} L, \tag{2.30}$$

where Γ and Ω are the matrices defined by (2.16) and (2.18), respectively.

Observe the following facts:

1. From (2.25) we get

$$\|\Phi^{(0)}\|_2 \leqslant \|L_0\|_2 \|\Gamma^{-1}\|_2 + \|K_0\|_2 \|\Omega^{-1}\|_2,$$

and from (2.30) we get

$$K^\dagger \Phi^{(0)} L^\dagger = \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix},$$

and

$$\max\left\{\|\Gamma^{-1}\|_2,\ \|\Omega^{-1}\|_2\right\} \leqslant \|K^\dagger\|_2 \|L^\dagger\|_2 \left\|\Phi^{(0)}\right\|_2.$$

2. By (1.9)–(1.11) we have

$$\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}) = \mathrm{sep}_{\mathrm{F}}(A_{11}^H, A_{22}^H) = \mathrm{sep}_{\mathrm{F}}(A_{22}, A_{11}).$$

Consequently, for the matrices Γ and Ω, the equality $\|\Gamma^{-1}\|_2 = \|\Omega^{-1}\|_2$ holds.

Hence, by (2.24) and (2.25) we have

$$\|K^\dagger\|_2^{-1} \|L^\dagger\|_2^{-1} \|\Gamma^{-1}\|_2 \leqslant c_{\mathrm{abs}}(P) \leqslant (\|K_0\|_2 + \|L_0\|_2) \|\Gamma^{-1}\|_2. \tag{2.31}$$

Remark 2.4. For the unitary matrix U of (1.1), let $U = (U_1, U_2)$, where $U_1 \in \mathscr{C}^{n\times m}$. By (1.3) we have $S_1 = U_1$, and from (1.1) (or (1.4)) we see that the subspace $\mathscr{R}(S_1)$ is the invariant subspace of A corresponding to $\lambda(A_{11})$. Let A be slightly perturbed to $A + \Delta A$, and let $\mathscr{S}_1 = \mathscr{R}(S_1)$ be perturbed to $\tilde{\mathscr{S}}_1$, correspondingly. By [6] we define the condition number $c(\mathscr{R}(S_1))$ of $\mathscr{R}(S_1)$ by

$$c(\mathscr{R}(S_1)) = \lim_{\delta\to 0} \sup_{\|\Delta A\|_{\mathrm{F}}\leqslant\delta} \frac{\rho_{\mathrm{F}}(\mathscr{S}_1, \tilde{\mathscr{S}}_1)}{\delta},$$

where $\rho_{\mathrm{F}}(\mathscr{S}_1, \tilde{\mathscr{S}}_1)$ is the generalized chordal metric defined by [7,8]

$$\rho_{\mathrm{F}}(\mathscr{S}_1, \tilde{\mathscr{S}}_1) = \frac{1}{\sqrt{2}} \|P_{\mathscr{S}_1} - P_{\tilde{\mathscr{S}}_1}\|_{\mathrm{F}},$$

in which $P_{\mathscr{S}_1}$ is the orthogonal projection onto $\mathscr{S}_1$. By [8, Chapter 2, Section 2.2], the condition number $c(\mathscr{R}(S_1))$ can be expressed by

$$c(\mathscr{R}(S_1)) = \|\Gamma^{-1}\|_2. \tag{2.32}$$

Combining it with (2.31) gives the relations

$$\|K^\dagger\|_2^{-1} \|L^\dagger\|_2^{-1} c(\mathscr{R}(S_1)) \leqslant c_{\mathrm{abs}}(P) \leqslant (\|K_0\|_2 + \|L_0\|_2) c(\mathscr{R}(S_1)), \tag{2.33}$$

where K_0, L_0, K and L are the matrices defined by (2.27)–(2.29). Note that for the matrices K_0 and L_0 we have

$$\|K_0\|_2 = 1 + \|M\|_2^2, \quad \|L_0\|_2 = 1 + \|M\|_2^2. \tag{2.34}$$

This fact can be proved as follows: Let $M = W\Sigma Q^H$ be a singular value decomposition of M, where W and Q are unitary matrices, and

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots), \quad \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant 0.$$

Then

$$\begin{aligned} K_0 &= \begin{pmatrix} I_m \\ -\overline{Q}\Sigma^{\mathrm{T}} W^{\mathrm{T}} \end{pmatrix} \otimes \begin{pmatrix} W\Sigma Q^H \\ I_{n-m} \end{pmatrix} \\ &= \left[\begin{pmatrix} \overline{W} & 0 \\ 0 & -\overline{Q} \end{pmatrix} \begin{pmatrix} I_m \\ \Sigma^{\mathrm{T}} \end{pmatrix} W^{\mathrm{T}} \right] \otimes \left[\begin{pmatrix} W & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \Sigma \\ I_{n-m} \end{pmatrix} Q^H \right] \\ &= \left[\begin{pmatrix} \overline{W} & 0 \\ 0 & -\overline{Q} \end{pmatrix} \otimes \begin{pmatrix} W & 0 \\ 0 & Q \end{pmatrix} \right] \left[\begin{pmatrix} I_m \\ \Sigma^{\mathrm{T}} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ I_{n-m} \end{pmatrix} \right] (W^{\mathrm{T}} \otimes Q^H), \end{aligned}$$

where the matrices

$$\begin{pmatrix} \overline{W} & 0 \\ 0 & -\overline{Q} \end{pmatrix} \otimes \begin{pmatrix} W & 0 \\ 0 & Q \end{pmatrix} \quad \text{and} \quad W^{\mathrm{T}} \otimes Q^H$$

are unitary. Consequently, we have

$$\|K_0\|_2 = \left\| \begin{pmatrix} I_m \\ \Sigma^{\mathrm{T}} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ I_{n-m} \end{pmatrix} \right\|_2 = 1 + \sigma_1^2 = 1 + \|M\|_2^2.$$

Similarly, we get the second equality of (2.34). Combining (2.34) with (2.33) gives an upper bound for $c_{\mathrm{abs}}(P)$:

$$c_{\mathrm{abs}}(P) \leqslant 2(1 + \|M\|_2^2) c(\mathscr{R}(S_1)) \equiv \beta(P). \tag{2.35}$$

Further, the relations (2.32) and (1.10) imply

$$c(\mathscr{R}(S_1)) = 1/\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}). \tag{2.36}$$

Moreover, from (1.2) we get

$$\mathrm{vec}\, M = -\Gamma^{-1} \mathrm{vec}\, A_{12},$$

and consequently,

$$\|M\|_2 \leqslant \|M\|_{\mathrm{F}} \leqslant \|\Gamma^{-1}\|_2 \|A_{12}\|_{\mathrm{F}} = \|A_{12}\|_{\mathrm{F}} / \mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22}). \tag{2.37}$$

Substituting (2.36) and (2.37) into (2.35) gives

$$c_{\mathrm{abs}}(P) \leqslant \frac{2}{\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22})} \left(1 + \frac{\|A_{12}\|_{\mathrm{F}}^2}{\mathrm{sep}_{\mathrm{F}}^2(A_{11}, A_{22})} \right). \tag{2.38}$$

Remark 2.5. By the theory of condition developed by Rice [6] we may define condition numbers of the generalized spectral projections associated with a regular matrix pair. Explicit expressions of the condition numbers are given by [9, Section 3].

3. A numerical example

We now use a simple numerical example to illustrate our results of Section 2. All computations were performed using MATLAB, version 6.5. The relative machine precision is 2.22×10^{-16}.

Example 3.1. Consider the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

with

$$A_{11} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \end{pmatrix},$$

and

$$A_{22} = \begin{pmatrix} 1-10^{-k} & 1 & 0 & 0 & 0 \\ 0 & 1-10^{-k} & 1 & 0 & 0 \\ 0 & 0 & 1-10^{-k} & 1 & 0 \\ 0 & 0 & 0 & 1-10^{-k} & 1 \\ 0 & 0 & 0 & 0 & 1-10^{-k} \end{pmatrix}.$$

Solving the Sylvester equation (1.2) we get M, and then substituting it into (1.3) and (1.5) gives S, T, and the spectral projection P of A corresponding to the eigenvalue 1. The computed condition numbers $c_{\mathrm{rel}}(P)$ and $c_{\mathrm{abs}}(P)$ (by (2.24)), the upper bound $\beta(P)$ for $c_{\mathrm{abs}}(P)$ (by (2.35)), the condition number $c(\mathscr{R}(S_1))$ of the invariant subspace $\mathscr{R}(S_1)$ (by (2.32)) and the quantity $\|M\|_2$ are listed in Table 1.

From the results listed in Table 1 we see that the spectral projection P is more sensitive to small changes in A when k increases. For understanding the results we point out the fact that for this example the separation $\mathrm{sep}_{\mathrm{F}}(A_{11}, A_{22})$ of A_{11} and A_{22} decreases with the increasing of k. Combining this fact with the relations (2.36)–(2.38) shows that $c(\mathscr{R}(S_1))$ increases and $\|M\|_2$ and $c_{\mathrm{abs}}(P)$ may increase with the increasing of k. Moreover, from the results listed in Table 1 we see that the upper bound $\beta(P)$ for $c_{\mathrm{abs}}(P)$ (see (2.35)) may be much larger than $c_{\mathrm{abs}}(P)$.

Table 1

k	-2	-1	0	1	2
$c_{\mathrm{rel}}(P)$	1.3	1.4	3.9×10^{2}	5.1×10^{9}	6.5×10^{16}
$c_{\mathrm{abs}}(P)$	1.0×10^{-2}	1.2×10^{-1}	1.1×10^{3}	1.2×10^{16}	1.9×10^{29}
$\beta(P)$	2.0×10^{-2}	2.4×10^{-1}	4.6×10^{3}	3.3×10^{22}	6.3×10^{41}
$c(\mathscr{R}(S_1))$	1.0×10^{-2}	1.2×10^{-1}	1.5×10	1.2×10^{8}	1.5×10^{15}
$\|M\|_2$	1.9×10^{-2}	2.0×10^{-1}	1.3×10	1.2×10^{7}	1.5×10^{13}

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