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# BOUNDS FOR THE DISTANCE BETWEEN NEARBY JORDAN AND KRONECKER STRUCTURES IN A CLOSURE HIERARCHY

**E. Elmroth, P. Johansson, and B. Kågström** UDC 519

*Computing the fine-canonical-structure elements of matrices and matrix pencils are ill-posed problems. Therefore, besides knowing the canonical structure of a matrix or a matrix pencil, it is equally important to know what are the nearby canonical structures that explain the behavior under small perturbations. Qualitative strata information is provided by our StratiGraph tool. Here, we present lower and upper bounds for the distance between Jordan and Kronecker structures in a closure hierarchy of an orbit or bundle stratification. This quantitative information is of importance in applications, e.g., distance to more degenerate systems (uncontrollability). Our upper bounds are based on staircase regularizing perturbations. The lower bounds are of Eckart-Young type and are derived from a matrix representation of the tangent space of the orbit of a matrix or a matrix pencil. Computational results illustrate the use of the bounds. Bibliography:* 42 *titles.*

**Dedicated to Vera N. Kublanovskaya**
**on the occasion of her 80th birthday**

## 1. Introduction

Computing the fine-canonical-structure elements of matrices and matrix pencils are ill-posed problems. Arbitrarily small perturbations in the data can drastically change the canonical structure, and the perturbations introduced by the finite-precision arithmetic are likely to corrupt the computed canonical forms. However, it is possible to regularize the problem by allowing a deflation criterion for range/null-space separations, and thereby to make it possible to compute the canonical structure of a nearby matrix or pencil. It was Vera Nikolaevna Kublanovskaya who laid the foundation in her 1966 paper [33], where she presented the *staircase algorithm* for computing the Jordan structure of a multiple eigenvalue by unitary similarity transformations. This milestone contribution stimulated numerous subsequent papers on algorithms for the numerical computation of Jordan and Kronecker structure information (e.g., see [40, 29, 28, 41, 26, 3, 11, 12, 30]), including several important contributions by Kublanovskaya herself (e.g., see [34, 35, 31, 32, 4, 36, 37]). Moreover, new insight and understanding of the mathematical theory of orbits and bundles of matrices and matrix pencils have led to new results, which, in turn, has stimulated the development of improved algorithms and new software tools (e.g., see [18, 14, 15, 17]).

Almost all $n \times n$ matrices have $n$ distinct eigenvalues and can be transferred into diagonal form by similarity transformations. This corresponds to the generic case. Only if the matrix lies in a particular manifold in the $n^2$-dimensional space of $n \times n$ matrices does it have a more interesting Jordan structure. The manifolds corresponding to all different structures form a *closure hierarchy*, i.e., a *stratification* of this space. The theory describing the complete stratification of orbits and bundles of matrices and matrix pencils is presented in [15], and a tool, StratiGraph, for computing and displaying closure hierarchies was recently presented in [17].

A stratification provides qualitative information about which structures are related to each other, which structures can be found near a specific matrix or matrix pencil, etc. Based on stratifications, this contribution provides quantitative results in terms of upper and lower bounds of the distance to the closest matrix or pencil with a specified structure. These quantitative results are important in applications, e.g., distance to more degenerate systems (uncontrollability). Our upper bounds are based on staircase regularizing perturbations. Our lower bounds are of Eckart–Young type and are derived from a matrix representation of the tangent space of the orbit of a matrix or a matrix pencil.

The outline of the paper is as follows. Section 2 introduces necessary definitions and notation for the matrix and matrix-pencil spaces, including canonical forms and structure characteristics and orbits and bundles. In Sec. 3, we discuss dimensions and codimensions of orbits and bundles and give some further details on stratifications. This section also includes an example presenting the complete stratification of orbits of $2 \times 4$ matrix pencils using StratiGraph [17]. Sections 4 and 5 present the theory underlying our lower and upper bounds, respectively. For the lower bounds, this includes the Kronecker product representation of the tangent space of the orbit of

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a matrix or a matrix pencil. For the upper bounds, we outline the main steps of the staircase algorithm and introduce the concept of regularizing perturbations in order to impose a specific Jordan or Kronecker structure. In Sec. 6, we present computational results and illustrate our bounds for orbits of $7 \times 7$ matrices and bundles of $3 \times 5$ matrix pencils. In Sec. 7, we summarize this contribution, present some related work, and give an outline for future work.

## 2. The matrix and matrix-pencil spaces

Any $n \times n$ matrix $A$ defines a manifold of similar matrices in the $n^2$-dimensional space $\mathcal{M}$ of square matrices. This manifold is the *similarity orbit* defined as

$$\mathcal{O}(A) = \{P^{-1}AP : \det(P) \neq 0\}.$$

We may choose a special element of $\mathcal{O}(A)$ that reveals the *Jordan canonical form* (JCF) of the matrix [19]:

$$AP = PJ,$$

where

$$J = \operatorname{diag}\{J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_t)\}.$$

Here, we assume that $A$ has $t$ distinct eigenvalues $\lambda_i$ with algebraic multiplicities $a_i$. The Jordan matrix $J(\lambda_i)$ is a direct sum of the Jordan blocks associated with the eigenvalue $\lambda_i$. The number of Jordan blocks for an eigenvalue is the same as its geometric multiplicity $g_i$. Let $s_k^{(i)}$ be the sizes of the Jordan blocks associated with $\lambda_i$, where $s_1^{(i)} \geq s_2^{(i)} \geq \ldots \geq s_{g_i}^{(i)} \geq 1$. Algebraically, the $s_k^{(i)}$s are the degrees of the elementary divisors of $A - \lambda I$ at $\lambda = \lambda_i$, also known as the *Segre characteristics*. We also see that $h_i = s_1^{(i)}$ is the maximum height of the vector chains for the eigenvalue $\lambda_i$. The Jordan matrix $J(\lambda_i)$ can be expressed as

$$J(\lambda_i) = \operatorname{diag}\{J_{s_1^{(i)}}(\lambda_i), J_{s_2^{(i)}}(\lambda_i), \ldots, J_{s_{g_i}^{(i)}}(\lambda_i)\}.$$

The complete Jordan matrix $J$ is uniquely defined up to the order of the Jordan blocks.

We also mention a dual way of characterizing the JCF. Let $w_k^{(i)}$ be the number of principal vectors of grade $k$ associated with $\lambda_i$ (or, equivalently, the number of $J_j(\lambda_i)$ blocks of size $j \geq k$). We have that $w_1^{(i)} \geq w_2^{(i)} \geq \ldots \geq w_{h_i}^{(i)} \geq 1$, which is the set of nonzero successive differences in the nullities of the matrices $(A - \lambda_i I)^k$ for $k = 1, \ldots, h_i$, also known as the *Weyr characteristics*.

The matrix-pencil analogue is to consider any $m \times n$ matrix pair $(A, B)$. Then $A - \lambda B$ defines an *orbit of strictly equivalent matrix pencils* in the $2mn$-dimensional space $\mathcal{P}$ of $m \times n$ pencils:

$$\mathcal{O}(A - \lambda B) = \{P^{-1}(A - \lambda B)Q : \det(P)\det(Q) \neq 0\}.$$

Similarly to the matrix case, we may choose a special element of $\mathcal{O}(A - \lambda B)$ that exhibits the fine-structure blocks of the matrix pencil, namely, the *Kronecker canonical form* (KCF) [19]. In addition to Jordan blocks for finite and infinite eigenvalues, the Kronecker form contains singular blocks corresponding to the minimal indices of a singular pencil.

Let $A, B \in \mathbf{C}^{m \times n}$. Then there exist nonsingular matrices $P \in \mathbf{C}^{m \times m}$ and $Q \in \mathbf{C}^{n \times n}$ such that

$$P^{-1}(A - \lambda B)Q = \tilde{A} - \lambda \tilde{B} \equiv \operatorname{diag}(A_1 - \lambda B_1, \ldots, A_b - \lambda B_b),$$

where $A_i - \lambda B_i$ is of size $m_i \times n_i$. Every block $M_i \equiv A_i - \lambda B_i$ must be of one of the following forms:

$$J_j(\alpha), \quad N_j, \quad L_j, \quad \text{or} \quad L_j^T.$$

The $J_j(\alpha)$ and $N_j$ are simply the Jordan blocks of the finite and infinite eigenvalues, respectively. These blocks together constitute the *regular structure* of the pencil.

The other two types of diagonal blocks are

$$L_j \equiv \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix} \quad \text{and} \quad L_j^T \equiv \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix}.$$

The $j \times (j+1)$ block $L_j$ is called a *singular block of right (or column) minimal index j*. It has a one-dimensional right null-space, $r_j = [1, \lambda, \dots, \lambda^j]^T$, such that $L_j r_j = 0$ for all $\lambda$. Similarly, the $(j+1) \times j$ block $L_j^T$ is a *singular block of left (or row) minimal index j*, and it has a one-dimensional left null-space for any $\lambda$. The left and right singular blocks together constitute the *singular structure* of the pencil and appear in the KCF if and only if the pencil is singular. The regular and singular structures define the *Kronecker structure* of a singular pencil. Also, the KCF is uniquely determined up to the order of the canonical blocks. For consistency reasons, the $L_j$ blocks appear before the $L_j^T$ blocks, while the Jordan blocks of the regular part may appear anywhere along the block diagonal of $\tilde{A} - \lambda \tilde{B}$.

Two elements of an orbit have exactly the same canonical structure, including the types and sizes of the blocks and the eigenvalues. A *bundle* is a union of orbits that have the same canonical structure, but their eigenvalues may differ. The concept of a bundle is defined both for matrices and matrix pencils, and we use the notation $\mathcal{B}(A)$ and $\mathcal{B}(A - \lambda B)$, respectively.

### 2.1. Generic and degenerate Jordan and Kronecker structures

Almost all $n \times n$ matrices have $n$ distinct eigenvalues. Hence, the generic Jordan structure is trivial; it consists of $n$ blocks $J_1$, each of which corresponds to a different eigenvalue. Only in the case of multiple eigenvalues is the Jordan structure more interesting. The most generic structure corresponding to an eigenvalue of multiplicity $k$ is $J_k$, a Jordan block of size $k \times k$. The Jordan structure of the most degenerate $n \times n$ matrix consists of $n$ blocks $J_1$, all of which correspond to the same eigenvalue (of multiplicity $n$).

The KCF looks quite complicated in the general case, but most matrix pencils have a more simple Kronecker structure. Almost all rectangular $m \times n$ pencils $A - \lambda B$ ($m \neq n$) have the same KCF, depending only on $m$ and $n$, and this KCF only includes the blocks $L_j$ if $m < n$ and $L_j^T$ otherwise (e.g., see [41, 10, 14]). This corresponds to the *generic case* where $A - \lambda B$ has full rank for any scalar $\lambda$. It follows that generic rectangular pencils have no regular part. We note that any pencil with only $L_j$ or only $L_j^T$ blocks has full rank, but it is only one of them which is the generic structure.

Square pencils are generically regular, i.e., $\det(A - \lambda B) = 0$ if and only if $\lambda$ is an eigenvalue. Moreover, the most generic regular pencil is diagonalizable and has distinct finite eigenvalues.

## 3. Hierarchies of nearby canonical structures

Orbits and bundles are manifolds in the $n^2$-dimensional space of $n \times n$ matrices and the $2mn$-dimensional space of $m \times n$ matrix pencils. The dimension of an orbit or bundle is uniquely determined by the Jordan or Kronecker structure. In practice, it is more convenient to work with the dimension of the space complementary to the orbit or bundle, called the *codimension*.

The normal space of an orbit or a bundle at a certain point (matrix or matrix pencil) is the space complementary and orthogonal to the tangent space at this point. The codimension is the dimension of the normal space, and it can be calculated from information about the Jordan and Kronecker structure (e.g., see [10]). It can also be computed as the number of zero singular values of a block matrix of Kronecker products [14] (see also Sec. 4).

The difference between the orbit and bundle cases is that, in the bundle case, we do not specify the value of an eigenvalue. Hence, for every eigenvalue in the Jordan or Kronecker form, the bundle has one more dimension compared to the corresponding orbit. It follows that the codimension count for this unspecified eigenvalue is one less. Therefore,

$$\text{cod}(\mathcal{B}(A)) = \text{cod}(\mathcal{O}(A)) - \#\text{ distinct eigenvalues of } A,$$

and

$$\text{cod}(\mathcal{B}(A - \lambda B)) = \text{cod}(\mathcal{O}(A - \lambda B)) - \#\text{ distinct eigenvalues of } A - \lambda B.$$

The codimensions for all generic and the most degenerate cases are summarized in Table 1. For example, for orbits of $n \times n$ matrices, the generic case has $\text{cod}(A) = n$ because there are $n$ specified and distinct eigenvalues. The corresponding most degenerate case has $\text{cod}(A) = n^2$ corresponding to a Jordan structure with $n$ blocks $J_1$ for the same eigenvalue. For example, if the eigenvalue is zero, then $A = 0_{n \times n}$.

The codimension counts induce a natural hierarchy: one matrix or pencil is more generic than another if it has lower codimension. However, this classification does not give the complete picture. Orbits and bundles of different canonical structures of a given size can have the same codimensions. We are also interested in knowing the relations between these and the other structures above and below in the hierarchy. The answer is given by a *stratified manifold*, which is the union of nonintersecting manifolds whose closures are finite unions of themselves with strata of smaller dimensions (thereby defining stratified manifolds recursively, see [2]). For matrices, the strata is a set of similarity orbits or bundles. For pencils, the strata is a set of equivalence orbits or bundles. We use the notation $\overline{\mathcal{O}}(\cdot)$ and $\overline{\mathcal{B}}(\cdot)$ to denote the closures of orbits and bundles, respectively. The *problem of stratification* is to find the closure relations among various orbits or bundles, i.e., the closure hierarchy of Jordan and Kronecker structures. These relations define a partial ordering on the set of orbits or bundles, which we call a *covering relationship.* One structure covers another if its closure contains the closure of the other and there is no other structure in between.

TABLE 1. Codimensions for the generic and the most degenerate orbits and bundles

| Dimensions | Orbits | Bundles | |
|---|---|---|---|
| $m = n$ | $\mathrm{cod}(A) = n$ | $\mathrm{cod}(A) = 0$ | |
| | $\mathrm{cod}(A - \lambda B) = n$ | $\mathrm{cod}(A - \lambda B) = 0$ | Generic |
| $m \neq n$ | $\mathrm{cod}(A - \lambda B) = 0$ | $\mathrm{cod}(A - \lambda B) = 0$ | |
| $m = n$ | $\mathrm{cod}(A) = n^2$ | $\mathrm{cod}(A) = n^2 - 1$ | Most |
| | $\mathrm{cod}(A - \lambda B) = 2mn$ | $\mathrm{cod}(A - \lambda B) = 2mn$ | degene- |
| $m \neq n$ | $\mathrm{cod}(A - \lambda B) = 2mn$ | $\mathrm{cod}(A - \lambda B) = 2mn$ | rate |

A stratification is a classification of all possible changes in the canonical structure that can take place for sufficiently small perturbations of a given matrix or pencil. Moreover, every possible change is smoothly attainable in terms of versal deformations (e.g., see [2] for the matrix case and [14] for general matrix pencils).

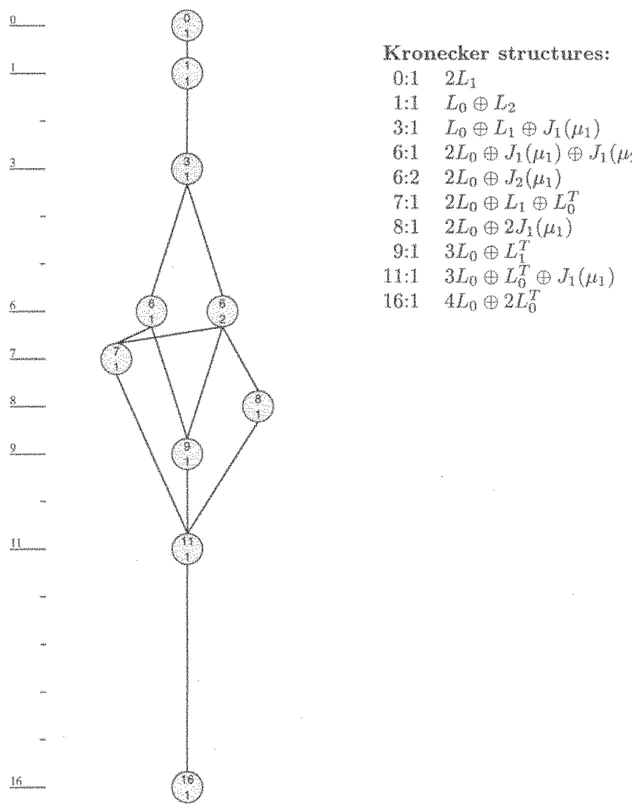


FIG. 1. The stratification of orbits of $2 \times 4$ matrix pencils.

A stratification can nicely be represented by a *graph.* A *node* represents an orbit or a bundle of matrices or matrix pencils. An *edge* represents a covering relation between two orbits or bundles, i.e., the one below is in the closure of the one above, and there is no other orbit or bundle in between. It follows that all orbits or bundles that can be reached by downward paths from a given node define orbits or bundles that are in the closure of the one represented by this node. A tool for computing and displaying stratifications of orbits and bundles of matrices and matrix pencils, called StratiGraph, has recently been presented in [17].

### 3.1. A sample stratification

We now illustrate this representation by considering Fig. 1, where StratiGraph presents the complete stratification of orbits of matrix pencils of size $2 \times 4$. The pencils live in a 16-dimensional space. The codimensions are given on the left-hand side of the window. The topmost node, marked "0 over 1," corresponds to the generic $2 \times 4$ pencil with KCF $2L_1$. This pencil has codimension 0, i.e., the closure of this orbit is the complete 16-dimensional space. It follows naturally that all other orbits are in $\overline{\mathcal{O}}(2L_1)$. In the graph, this is represented by *paths* downward from "0 over 1" to every other node.

The second most generic structure, $L_0 \oplus L_2$ with codimension 1, is represented by the node "1 over 1" immediately below the generic structure. Since the codimension is 1, this orbit is a 15-dimensional space. The node "3 over 1," representing $\mathcal{O}(L_0 \oplus L_1 \oplus J_1(\mu_1))$, defines a 13-dimensional space in the closure of $\mathcal{O}(L_0 \oplus L_2)$. Downward from this node, we have two edges to "6 over 1" ($\mathcal{O}(2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2))$) and "6 over 2" ($\mathcal{O}(2L_0 \oplus J_2(\mu_1))$). These orbits with codimension 6 represent two independent 10-dimensional spaces.

The nodes "7 over 1" ($\mathcal{O}(2L_0 \oplus L_1 \oplus L_0^T)$) and "9 over 1" ($\mathcal{O}(3L_0 \oplus L_1^T)$) define two independent spaces (of dimension 9 and 7, respectively) in the intersection of the spaces represented by "6 over 1" and "6 over 2." The node "8 over 1" represents $\mathcal{O}(2L_0 \oplus 2J_1(\mu_1))$, an 8-dimensional space, also in $\overline{\mathcal{O}}(2L_0 \oplus J_2(\mu_1))$.

An interesting case is $\mathcal{O}(3L_0 \oplus L_0^T \oplus J_1(\mu_1))$, the node "11 over 1." It is a 5-dimensional space in the intersection of the closures of the 9-, 8-, and 7-dimensional spaces represented by the nodes "7 over 1," "8 over 1," and "9 over 1."

Finally, in the closure of all other orbits, we find the bottommost node "16 over 1," representing $\mathcal{O}(4L_0 \oplus 2L_0^T)$, which corresponds to $A = B = 0$. The codimension of the orbit is 16, i.e., it is a point in the 16-dimensional space of $2 \times 4$ matrix pencils.

Since all other structures can be found by following paths upward from this orbit, the zero pencil is in the closure of every other orbit. This is of course also easy to verify because any Kronecker structure can be imposed by a small perturbation of the zero pencil.

## 4. Lower bounds for the distances to less generic matrices and pencils

In [14], Edelman, Elmroth, and Kågström presented a general technique for deriving lower bounds for the distances to less generic matrices and matrix pencils. It is based on a matrix representation of the tangent space of a matrix or matrix-pencil orbit and leads to Eckart–Young-type bounds expressed in terms of the singular values of the "tangent space matrix." We use these bounds for obtaining lower bounds for the distance to nearby Jordan and Kronecker structures in a closure hierarchy. For completeness, below we give an overview of the results, focusing on the matrix-pencil case.

The dimension of $\mathcal{O}(A - \lambda B)$ is equal to the dimension of the space tangent to the orbit at $A - \lambda B$, which we denote by $\tan(A - \lambda B)$. The tangent space of $\mathcal{O}(A - \lambda B)$ consists of pencils represented in the form

$$T_A - \lambda T_B = (XA - AY) - \lambda(XB - BY),$$

where $X$ is an $m \times m$ matrix and $Y$ is an $n \times n$ matrix.

Using Kronecker products, we can represent $T_A - \lambda T_B \in \tan(A - \lambda B)$ as the $2mn \times (m^2 + n^2)$ matrix

$$T \equiv \begin{bmatrix} A^T \otimes I_m & -I_n \otimes A \\ B^T \otimes I_m & -I_n \otimes B \end{bmatrix}. \tag{1}$$

Moreover, it follows that the range of $T$ represents a basis for the tangent space of $A - \lambda B$:

$$\tan(A - \lambda B) = \operatorname{range}(T) = \{T_A - \lambda T_B\}.$$

Since the space orthogonal to the range of a matrix is the kernel of the Hermitian transpose, we also have

$$\operatorname{nor}(A - \lambda B) = \ker(T^H) = \ker \begin{bmatrix} \bar{A} \otimes I_m & \bar{B} \otimes I_m \\ -I_n \otimes A^H & -I_n \otimes B^H \end{bmatrix}. \tag{2}$$

These facts lead to the following "compact" characterization of the codimension of the orbit $\mathcal{O}(A-\lambda B)$ [14]:

$$\operatorname{cod}\mathcal{O}(A-\lambda B) = \text{the number of zero singular values of } T(1). \tag{3}$$

The matrix representation of the tangent and normal spaces and the *SVD* characterization of the codimension of $\mathcal{O}(A-\lambda B)$ lead to the following information on the distance between nearby structures in a closure hierarchy.

For a given $m \times n$ pencil $A-\lambda B$ with codimension $c$, we derive the following lower bound for the distance to the *closest pencil* $C-\lambda D$ *with codimension* $c+d$, where $d \geq 1$ [14]:

$$\|(A-C, B-D)\|_E \geq \frac{1}{\sqrt{m+n}} \left( \sum_{i=2mn-c-d+1}^{2mn} \sigma_i^2(T) \right)^{1/2}. \tag{4}$$

Here, $\sigma_i(T)$ denotes the $i$th largest singular value of $T$ ($\sigma_i(T) \geq \sigma_{i+1}(T) \geq 0$).

Starting with the generic case, i.e., $\operatorname{cod}\mathcal{O}(A-\lambda B) = c = 0$, we can traverse the closure-hierarchy graph and obtain information on distances to all less generic structures in the stratification. As a special case, we obtain a lower bound for the distance to the *closest degenerate pencil* $C-\lambda D$:

$$\|(A-C, B-D)\|_E \geq \frac{\sigma_{\min}(T)}{\sqrt{m+n}}, \tag{5}$$

where $\sigma_{\min}(T) = \sigma_{2mn}(T)$ is the smallest nonzero singular value of $T$ in (1).

If $A-\lambda B$ is a square $n \times n$ regular pencil, then (5) gives a lower bound for the distance to the *closest nonregular (or singular) pencil*. Other types of bounds on the closest nonregular pencil are presented in [8].

Another application is to estimate the *distance to uncontrollability* for a multiple-input-multiple-output linear system $E\dot{x}(t) = Fx(t) + Gu(t)$, where $E$ and $F$ are $m \times m$ matrices, $G$ is $m \times p$ ($m \geq p$), and $E$ is assumed to be nonsingular. If $A-\lambda B \equiv [G|F-\lambda E]$ is generic, the linear system is controllable (i.e., the dimension of the controllable subspace equals $m$), and a lower bound for the distance to the closest uncontrollable system is given by (5) (see Sec. 6.2 for some computational results).

Similar bounds are, of course, valid for the matrix case. The matrix representation of the tangent space of an $n \times n$ matrix $A$ in $\mathcal{O}(A)$ is $T = I_n \otimes A - A^T \otimes I_n$ of size $n^2 \times n^2$. Now, for a given $A$ with codimension $c$, we obtain the following lower bound for the distance to the *closest matrix* $C$ *with codimension* $c+d$, where $d \geq 1$:

$$\|A-C\|_E \geq \frac{1}{\sqrt{2n}} \left( \sum_{i=n^2-c-d+1}^{n^2} \sigma_i^2(T) \right)^{1/2}. \tag{6}$$

## 5. Upper bounds for the distances to less generic matrices and pencils

The *staircase algorithm* due to Vera N. Kublanovskaya (1966) [33] is our basic tool for computing upper bounds for the distances between nearby canonical structures in a closure hierarchy. Our bounds are based on staircase regularizing perturbations, which means that we apply a regularization technique by allowing a deflation criterion for range/null-space separations in finite-precision arithmetic, thereby making it possible to compute or impose the canonical structure of a nearby problem.

### 5.1. The Schur-staircase forms

In general, we cannot guarantee that the JCF of a matrix or the KCF of a pencil is computed stably because the transformation matrices that reduce $A$ to JCF or $A-\lambda B$ to KCF can be arbitrarily ill-conditioned. However, it is possible to compute the Jordan/Kronecker structure (or parts of it) by using only unitary transformations. The price we must pay is a denser canonical form, called the *Schur-staircase form* in the matrix case and the *generalized Schur-staircase form* for matrix pencils. These forms are block upper triangular with diagonal blocks in staircase form (also block upper triangular) that reveal the fine-structure elements of the JCF and KCF, respectively. We illustrate this with the most general case.

In most applications, it is sufficient to reduce $A-\lambda B$ to the generalized Schur-staircase form, e.g., to the *GUPTRI form* [11, 12]

$$P^H(A-\lambda B)Q = S - \lambda T \equiv \begin{bmatrix} A_r - \lambda B_r & * & * \\ 0 & A_{\text{reg}} - \lambda B_{\text{reg}} & * \\ 0 & 0 & A_l - \lambda B_l \end{bmatrix}, \tag{7}$$

where $P$ $(m \times m)$ and $Q$ $(n \times n)$ are unitary, and $*$ denotes arbitrary conforming submatrices. Here, the square upper triangular block $A_{\mathrm{reg}} - \lambda B_{\mathrm{reg}}$ is regular and has the same regular structure as $A - \lambda B$ (i.e., it contains all finite and infinite eigenvalues of $A - \lambda B$). The rectangular block $A_r - \lambda B_r$ has only right minimal indices in its KCF, actually the same blocks $L_j$ as $A - \lambda B$. Similarly, $A_l - \lambda B_l$ has only left minimal indices in its KCF, the same blocks $L_j^T$ as $A - \lambda B$. If $A - \lambda B$ is singular, at least one of the blocks $A_r - \lambda B_r$ and $A_l - \lambda B_l$ will be present in (7). If $A - \lambda B$ is regular, then neither $A_r - \lambda B_r$ nor $A_l - \lambda B_l$ is present in (7), and the GUPTRI form reduces to $A_{\mathrm{reg}} - \lambda B_{\mathrm{reg}}$. Staircase forms that reveal the Jordan structure of the zero and infinite eigenvalues are contained in $A_{\mathrm{reg}} - \lambda B_{\mathrm{reg}}$:

$$A_{\mathrm{reg}} = \begin{bmatrix} A_z & * & * \\ 0 & A_f & * \\ 0 & 0 & A_i \end{bmatrix}, \qquad B_{\mathrm{reg}} = \begin{bmatrix} B_z & * & * \\ 0 & B_f & * \\ 0 & 0 & B_i \end{bmatrix}. \tag{8}$$

In summary, the diagonal blocks of the GUPTRI form of $A - \lambda B$ describe the Kronecker structure as follows:

$A_r - \lambda B_r$ reveals the whole right singular structure (the right minimal indices);
$A_z - \lambda B_z$ reveals the Jordan structure for the zero eigenvalue;
$A_f - \lambda B_f$ contains all finite, nonzero eigenvalues;
$A_i - \lambda B_i$ reveals the Jordan structure for the infinite eigenvalue;
$A_l - \lambda B_l$ reveals the whole left singular structure (the left minimal indices).

For a description of the explicit structure of the diagonal blocks in staircase form, we refer the reader to [11, 12, 27]. The nonzero finite eigenvalues of $A - \lambda B$ (if any) are in the block $A_f - \lambda B_f$, but their multiplicities or Jordan structures are not computed explicitly. However, it is possible to extract the Jordan structure of a finite nonzero eigenvalue of $A - \lambda B$ by computing the $RZ$-staircase form[1] of the shifted pencil $(A_f - \lambda_i B_f) - \lambda B_f$, which has zero as an eigenvalue of multiplicity $\geq 1$.

The deflations made in the staircase algorithm result in the *exact GUPTRI form of a nearby matrix pencil* $A + \delta A - \lambda(B + \delta B)$:

$$P^H(A + \delta A - \lambda(B + \delta B))Q = S - \lambda T.$$

If all the range and null-space separations in the GUPTRI algorithm are well defined with respect to the deflation-tolerance parameters (Tol and Gap)[2], then $\|(\delta A, \delta B)\|_F = \mathcal{O}(\text{Tol}\|(A, B)\|_F)$. The nearby problem $C - \lambda D \equiv P(S - \lambda T)Q^H$ represents a regularized problem that has a stable Kronecker structure with respect to the deflation criteria of the algorithm.

### 5.2. Upper bounds via computing or imposing a canonical structure

By applying the standard (or greedy) staircase algorithm as described in the previous section, we immediately obtain an upper bound for the distance to the *nearest matrix pencil with the computed Kronecker structure* as the exact one:

$$\|(A - C, B - D)\|_F \leq \|(\delta A, \delta B)\|_F, \tag{9}$$

where $\|(\delta A, \delta B)\|_F^2$ is the sum of squares of "deleted" singular values in the deflation steps. Note that there might exist a closer $\tilde{C} - \lambda \tilde{D}$ in $\mathcal{O}(S - \lambda T)$. Minimization techniques for computing a closer $\tilde{C} - \lambda \tilde{D}$ are discussed in [13, 38].

We have also designed different variants of the staircase algorithm, where we impose a canonical structure (or staircase form). Starting with the generic case, our objective is to traverse the closure-hierarchy graph by imposing all possible structures. Let $A - \lambda B$ be a pencil with a known canonical structure. Then we compute unitary transformation matrices $P$ and $Q$ such that

$$P^H(A + \delta A - \lambda(B + \delta B))Q = S_{\mathrm{imp}} - \lambda T_{\mathrm{imp}},$$

where the generalized Schur-staircase form $S_{\mathrm{imp}} - \lambda T_{\mathrm{imp}}$ has the imposed canonical structure.

Several deficiencies can appear during such an application of an "imposing" staircase algorithm. We illustrate this with the nilpotent matrix case.

[1] The $RZ$-staircase form ($RZ$ for Right-Zero) reveals the right singular structure and the Jordan structure of the zero eigenvalue of $A - \lambda B$.

[2] In order to determine which singular values are to be considered zero, the two parameters Tol and Gap are used. Singular values smaller than Tol are considered zero, and the smallest nonzero singular value must be at least a factor Gap larger than the largest singular value that is considered zero.

### 5.3. Imposed staircase forms. The nilpotent case

Without loss of generality, we assume that $n = 7$ and fix a Jordan structure, e.g., $J_4(0) \oplus J_2(0) \oplus J_1(0)$, to be imposed. Given an arbitrary $n \times n$ nilpotent matrix $A$, the corresponding *imposed Schur-staircase form* looks like

$$Q^H(A + QEQ^H)Q = S_{\text{imp}},$$

where

$$S_{\text{imp}} = \begin{bmatrix} 0 & S_{12} & S_{13} \\ 0 & 0 & S_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix}.$$

The superdiagonal blocks $S_{ii+1}$ are of size $m_i \times m_{i+1}$ with $m_1 = 3, m_2 = 2$, and $m_3 = 2$. Of course, we obtain the same block structure of $S_{\text{imp}}$ as if $A$ would have had the Jordan structure that we imposed. If each of the blocks $S_{ii+1}$ is rank deficient, then $A$ has a more degenerate (less generic) structure in $\overline{\mathcal{O}}(S_{\text{imp}})$, which can be obtained by applying the standard staircase algorithm to $A$.

That is fine, but what can we do if we really want to impose a given Jordan structure? One answer is that we can force the blocks $S_{ii+1}$ to be of full column rank by adding a *staircase regularizing perturbation* to $S_{\text{imp}}$:

$$Q^H(A + QEQ^H + QFQ^H)Q = S_{\text{imp}} + F,$$

where

$$F = \begin{bmatrix} 0 & F_{12} & 0 \\ 0 & 0 & F_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $F$ is a (staircase) perturbation in $\overline{\mathcal{O}}(S_{\text{imp}})$. In theory, $\|F\|_F$ can be chosen arbitrarily small. In finite-precision arithmetic, the size of $\|F\|_F$ must be at least of the size of the tolerance parameter used for the rank decisions. Otherwise, we identify a less generic structure!

Our procedure for determining $F$ is as follows. Assume that $S_{ii+1}$ of size $m_i \times m_{i+1}$ has nullity $c \geq 0$. From the SVD of $S_{ii+1} (= U\Sigma V^H)$ we compute a rank-$c$ perturbation

$$F_{ii+1} = U[:, 1:c] D V[:, 1:c]^H \tag{10}$$

with $D = \text{diag}(d_i) > 0$. Since we want $\|F\|$ to be as small as possible, we choose $d_i$ with respect to the deflation criterion ($=$ Tol $\cdot$ Gap).

Our upper bound for the distance from $A$ to the nearby nilpotent $C = Q(S_{\text{imp}} + F)Q^H$ with the imposed (requested) Jordan structure is

$$\|A - C\|_F \leq \|E\|_F + \|F\|_F. \tag{11}$$

## 6. Examples and computational results

We illustrate the use of our bounds with two examples. The first is a nilpotent orbit, and we discuss distance information concerning a given generic matrix and all structures in the closure hierarchy. In the second example, we consider a bundle of matrix pencils and investigate a substratification associated with the "controllability hierarchy" of linear systems $Ex'(t) = Fx(t) + Gu(t)$.

### 6.1. A nilpotent orbit

The stratification of the nilpotent $7 \times 7$ orbits and the associated closure-hierarchy graph are displayed in Fig. 2. The topmost node corresponds to the generic case, i.e., one $7 \times 7$ Jordan block associated with the zero eigenvalue. The bottom node corresponds to the most degenerate case, i.e., seven Jordan blocks $J_1(0)$ (the zero matrix), and the orbit corresponds to a point in the 49-dimensional space of $7 \times 7$ nilpotent matrices (cod $= 49$).

In the sequel, we consider a matrix $A = ZJZ^T$ in $\mathcal{O}(J_4(0) \oplus J_2(0) \oplus J_1(0))$, where

$$J = \text{diag}\left( \begin{bmatrix} 0 & 1.00e-9 & 0 & 0 \\ 0 & 0 & 2.15e-10 & 0 \\ 0 & 0 & 0 & 4.64e-11 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1.00e-11 \\ 0 & 0 \end{bmatrix}, [\,0\,] \right)$$

and $Z$ is a $7 \times 7$ random orthogonal matrix. The Segre and Weyr characteristics of $A$ are [4, 2, 1] and [3, 2, 1, 1], respectively.

For $\text{Tol} = 2.204e-15$ and $\text{Gap} = 500$, the greedy staircase algorithm delivers the following Schur-staircase form:

$$S=\left[\begin{array}{ccc|cc|c|c}
0 & 0 & 0 & 7.35e{-10} & -3.36e{-11} & -1.16e{-24} & 4.41e{-25}\\
0 & 0 & 0 & -2.64e{-10} & 1.70e{-11} & 4.24e{-25} & -1.65e{-25}\\
0 & 0 & 0 & -6.22e{-10} & 4.11e{-11} & 9.03e{-25} & -3.60e{-25}\\
\hline
0 & 0 & 0 & 0 & 0 & -2.15e{-10} & 1.42e{-24}\\
0 & 0 & 0 & 0 & 0 & 1.18e{-11} & -5.85e{-26}\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -4.64e{-11}\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

When we try to impose the *covered structure* $2J_3 \oplus J_1$ (with Weyr characteristics $[3,2,2]$) on $A$, we arrive at the following results returned by the imposed staircase algorithm (with no rank checks on $S_{i+1}$):

$$S_{\text{imp}}=\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 7.35e{-10} & -3.36e{-11} & 4.41e{-25} & -1.16e{-24}\\
0 & 0 & 0 & -2.64e{-10} & 1.70e{-11} & -1.65e{-25} & 4.24e{-25}\\
0 & 0 & 0 & -6.22e{-10} & 4.11e{-11} & -3.60e{-25} & 9.03e{-25}\\
\hline
0 & 0 & 0 & 0 & 0 & 1.42e{-24} & -2.15e{-10}\\
0 & 0 & 0 & 0 & 0 & -5.85e{-26} & 1.18e{-11}\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

By inspection, we see that the superdiagonal block (2,3) of $S_{\text{imp}}$ does not have full column rank with respect to the rank-deflation tolerance $\text{Tol}$. Indeed, $S_{\text{imp}}$ has a null-space of dimension four, and the computed Schur-staircase form corresponds to the less generic structure $J_3 \oplus J_2 \oplus 2J_1$ with Weyr characteristics $[4,2,1]$. The lower and upper bounds for the distance to a nilpotent matrix with this structure are $1.8707e-11$ and $4.6416e-11$, respectively.

Upon adding the staircase regularizing perturbation

$$F_{23} = \begin{bmatrix} 1.22e-13 & 0\\ 2.22e-12 & 0 \end{bmatrix},$$

computed in accordance with (10), the greedy staircase algorithm applied to $S+F$ produces the following Schur-staircase form, corresponding to the desired imposed structure $2J_3 \oplus J_1$:

$$S=\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 3.87e{-10} & -5.78e{-12} & 8.62e{-25} & -2.51e{-22}\\
0 & 0 & 0 & -8.35e{-10} & 3.44e{-11} & -1.69e{-24} & 5.44e{-22}\\
0 & 0 & 0 & 3.89e{-10} & -1.84e{-11} & 7.48e{-25} & -2.53e{-22}\\
\hline
0 & 0 & 0 & 0 & 0 & 2.15e{-10} & -1.25e{-13}\\
0 & 0 & 0 & 0 & 0 & -8.23e{-12} & -2.22e{-12}\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

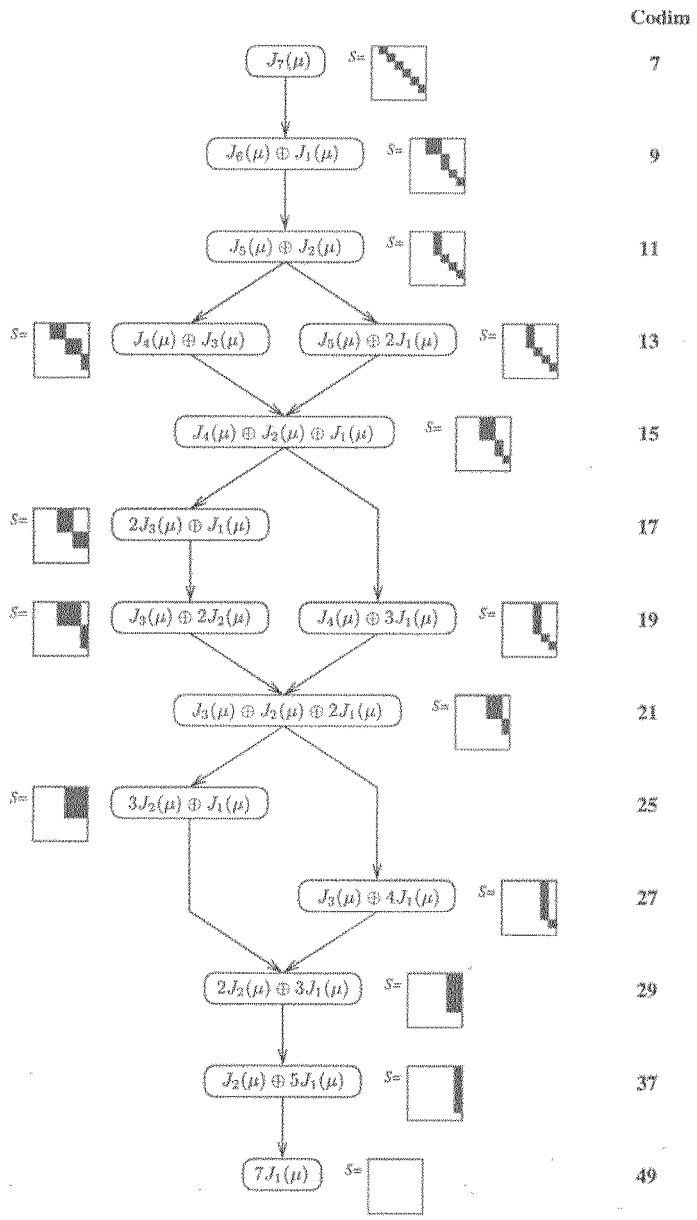


FIG. 2. The closure-hierarchy graph of $\overline{\mathcal{O}}(J_7(0))$ along with the Schur-staircase forms $S$ associated with each of the Jordan structures. The black boxes in every $S$ correspond to the superdiagonal blocks $S_{ii+1}$ of full column rank.

Now, all the blocks $S_{ii+1}$ have full column rank with respect to TOL. Moreover, the lower and upper bounds for the distance to the imposed structure are $3.7796e-12$ and $4.6469e-11$, respectively. The ratio between the bounds has increased because we have added fewer nonzero singular values in the lower bound (the imposed structure has lower codimension), whereas the upper bound is of the same size (we have only added a "zero-sized" perturbation $F$).

In Table 2, we present lower and upper bounds for the distance between $A$ and the closest nilpotent matrices in $\overline{\mathcal{O}}(J_4(0) \oplus J_2(0) \oplus J_1(0))$ (see Fig. 2). The possible Jordan structures are marked with a "right-arrow" ($\longrightarrow$) followed by the corresponding Segre characteristic. In five cases, we first found more degenerate structures. Each of them is listed just below the structure we wanted to impose, and, in these cases, we added staircase regularizing perturbations as discussed above.

Table 2. Distance information for imposed structures (Segre characteristics) in $\overline{\mathcal{O}}(J_4(0) \oplus J_1(0))$ with and without staircase regularization always outgoing from $\mathcal{O}(J_4(0) \oplus J_2(0) \oplus J_1(0))$

| From | To | cod | Lower bound | Upper bound | Ratio |
|---|---|---|---|---|---|
| [421] | | 15 | | | |
| $\longrightarrow$ | [331] | 17 | $3.78e-12$ | $4.65e-11$ | 12.29 |
| | [3211] | 21 | $1.87e-11$ | $4.64e-11$ | 2.48 |
| $\longrightarrow$ | [322] | 19 | $6.55e-12$ | $4.65e-11$ | 7.10 |
| | [3211] | 21 | $1.87e-11$ | $4.64e-11$ | 2.48 |
| $\longrightarrow$ | [4111] | 19 | $6.55e-12$ | $1.00e-11$ | 1.53 |
| $\longrightarrow$ | [3211] | 21 | $1.87e-11$ | $4.75e-11$ | 2.54 |
| | [31111] | 27 | $1.87e-11$ | $4.64e-11$ | 2.48 |
| $\longrightarrow$ | [2221] | 25 | $3.13e-11$ | $2.21e-10$ | 7.05 |
| | [211111] | 37 | $1.95e-10$ | $2.21e-10$ | 1.13 |
| $\longrightarrow$ | [31111] | 27 | $3.72e-11$ | $4.75e-11$ | 1.28 |
| $\longrightarrow$ | [22111] | 29 | $8.95e-11$ | $2.21e-10$ | 2.46 |
| | [211111] | 37 | $1.95e-10$ | $2.21e-10$ | 1.13 |
| $\longrightarrow$ | [211111] | 37 | $1.95e-10$ | $2.21e-10$ | 1.13 |
| $\longrightarrow$ | [1111111] | 49 | $1.02e-09$ | $1.02e-09$ | 1.00 |

### 6.2. A pencil bundle stratification with applications to control theory

The stratification of bundles of $3 \times 5$ matrix pencils is displayed in Fig. 3.

This is a stratification of a 30-dimensional space with, in total 26, different Kronecker structures. The most generic structure, $L_1 \oplus L_2$ with codimension 0, is the topmost node of the closure-hierarchy graph.

This stratification is of interest when studying controllability issues for the linear system

$$Ex'(t) = Fx(t) + Gu(t),$$

where $E$ and $F$ are $m \times m$ and $G$ is $m \times p$. The system is *controllable* if, starting with $x(0) = x_0$, it is possible to choose an input $u$ to bring the state vector $x$ to an arbitrary state in some finite time $t_N$. One way of characterizing controllability is via the controllability pencil

$$\mathcal{C}(E, F, G) = [G|F] - \lambda[0|E].$$

It has full rank except at $k < m$ values of $\lambda$ that correspond to the *uncontrollable modes* of the linear system above.

The stratification in Fig. 3 is for the controllability pencil of a linear system with three states ($m = 3$) and two inputs ($p = 2$). The graph gives the complete picture, but since $E$ is assumed to be nonsingular, several of the Kronecker structures represented in the graph are not possible for this application. In the sequel, we only consider the part of the stratification that is relevant to this application, i.e., we do not consider structures that include blocks $L_j^T$ or infinite eigenvalues.

Let us investigate the topmost nodes of the stratification graph. Next to the generic case $L_1 \oplus L_2$ there are two structures with codimension 2, namely, $L_0 \oplus L_3$ and $2L_1 \oplus J_1(\mu_1)$. Each of these two bundles is a 28-dimensional space. All other bundles are found in the intersection of these two spaces.

The most degenerate case considered here is $2L_0 \oplus 3J_1(\mu_1)$ (denoted 14:1 in the graph). The dimension of its bundle is 16 ($= 30 - 14$).

Table 3 presents upper and lower bounds for the perturbations needed to find pencils with prescribed Kronecker structures. The results are the mean values obtained from 100 random pencils, normalized so that $\|A\|_F = 1$ and $\|B\|_F = 1$. For every random pencil, the lower bounds are computed using (4) in Sec. 4, and the upper bounds are obtained by imposing a pencil with prescribed structure by applying the modified GUPTRI software, as described in Sec. 5.

In this example, the values of the rank-deflation parameters used are Tol $= 1e-10$ and Gap $= 1000$.

Table 3. Upper and lower bounds for the distance from a random generic pencil to pencils with prescribed Kronecker structures (mean values for 100 random problems)

| Imposed Structure | cod | Lower bound | Upper bound | Ratio |
|---|---|---|---|---|
| $L_0 \oplus L_3$ | 2 | $1.63e-02$ | $1.31e-01$ | 8.07 |
| $2L_1 \oplus J1(\mu_1)$ | 2 | $1.63e-02$ | $1.38e-01$ | 8.49 |
| $L_0 \oplus L_2 \oplus J1(\mu_1)$ | 3 | $2.43e-02$ | $1.57e-01$ | 6.45 |
| $L_0 \oplus L_1 \oplus J1(\mu_1) \oplus J1(\mu_2)$ | 4 | $3.93e-02$ | $2.19e-01$ | 5.57 |
| $L_0 \oplus L_1 \oplus J2(\mu_1)$ | 5 | $5.30e-02$ | $4.71e-01$ | 8.89 |
| $2L_0 \oplus J1(\mu_1) \oplus J1(\mu_2) \oplus J1(\mu_3)$ | 6 | $6.58e-02$ | $2.64e-01$ | 4.02 |
| $2L_0 \oplus J2(\mu_1) \oplus J1(\mu_2)$ | 7 | $8.37e-02$ | $3.44e-01$ | 4.11 |
| $L_0 \oplus L_1 \oplus 2J1(\mu_1)$ | 7 | $8.37e-02$ | $3.55e-01$ | 4.24 |
| $2L_0 \oplus J3(\mu_1)$ | 8 | $1.05e-01$ | $5.27e-01$ | 5.01 |
| $2L_0 \oplus 2J1(\mu_1) \oplus J1(\mu_2)$ | 9 | $1.25e-01$ | $3.77e-01$ | 3.02 |
| $2L_0 \oplus J1(\mu_1) \oplus J2(\mu_1)$ | 10 | $1.46e-01$ | $5.42e-01$ | 3.72 |
| $2L_0 \oplus 3J1(\mu_1)$ | 14 | $2.46e-01$ | $9.93e-01$ | 4.04 |

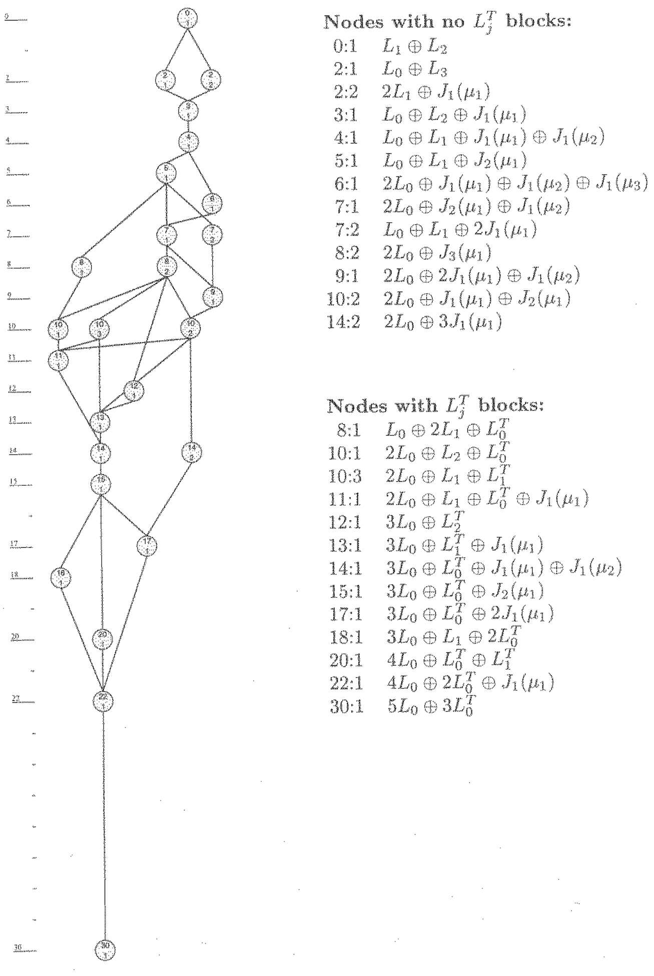


Fig. 3. The stratification of bundles of $3 \times 5$ matrix pencils.

In the standard version of the GUPTRI software as well as the version we use to impose specified structures, the zero (and infinite) eigenvalue and the nonzero finite eigenvalues are treated differently. Since a zero eigenvalue corresponds to the null-space of $A$, the structure for this eigenvalue is determined in connection with the range and null-space separations performed in order to determine the right singular structure (the blocks $L_j$). The structure for the nonzero finite eigenvalues is computed upon reducing the pencil to a square regular pencil without zero and infinite eigenvalues.

When imposing a specific structure in order to find the corresponding bundle, we do not want to specify whether an eigenvalue is zero or not because both alternatives would place us in the same bundle. In our striving to find the closest pencil in that bundle we, therefore, impose both cases, i.e., one with an eigenvalue specified as zero and one without. As the result we choose the one we find to be the closest.

In addition to the mean values of the bounds, Table 3 also reports the ratio of the upper bound to the lower bound. The ratio is a measure of how close we are able to bound the distance relative to the actual distance.

Since the tests are performed on random pencils, we would typically not find nongeneric structures at very small distances. In Table 3, all lower bounds reported are in the range from $1.6e-2$ to $2.4e-1$, and the upper bounds are in the range from $1.3e-1$ to $9.9e-1$. For all cases, the ratios of the bounds are in the range between 3.0 and 8.9. Since the exact distance is unknown for most cases, the ratio is our best measure for the quality of the bounds. A high ratio means that one or both of the bounds are far from the exact value, but with all ratios less than ten we conclude that all bounds are fairly tight.

For the two cases $L_0 \oplus L_3$ and $2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_2)$, the distance to the closest pencil is known to be the smallest perturbation that reduces the rank of the $6 \times 5$ matrix

$$\begin{bmatrix} A \\ B \end{bmatrix} \tag{12}$$

by one and two, respectively [18]. The Frobenius norms of the perturbations required are $\sigma_1$ and $\sqrt{\sigma_1^2 + \sigma_2^2}$, respectively, where $\sigma_1 \leq \sigma_2$ are the two smallest singular values of (12).

For the 100 test problems, the average distance for $L_0 \oplus L_3$ is $8.91e-2$. Our upper bound ($1.31e-1$) is a factor 1.48 larger than the actual distance, and the lower bound ($1.63e-2$) is a factor 5.47 smaller than the actual distance. For $2L_0 \oplus J_1(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_2)$, the upper bound ($2.64e-1$) is a factor 1.23 larger than the actual distance, and the lower bound ($6.58e-2$) is a factor 3.26 smaller than the actual distance.

## 7. Summary, related and future work

We have presented lower and upper bounds for the distance between Jordan and Kronecker structures in a closure hierarchy of an orbit or bundle stratification. This quantitative information is of importance in applications, e.g., distance to more degenerate systems (uncontrollability). Our upper bounds are based on staircase regularizing perturbations. The lower bounds are of Eckart–Young type and are derived from a matrix representation of the tangent space of the orbit of a matrix or a matrix pencil. We have also presented computational results that illustrate the use and reliability of the bounds. For the examples considered, the ratios between the upper and lower bounds are very sharp (less than ten).

Currently, qualitative strata information is provided by our StratiGraph tool [17, 25], which is built on the mathematical theory of stratification of orbits and bundles of matrices and matrix pencils that was recently completed (see [15]). Earlier works on the problem of stratification for matrices and matrix pencils include those of Arnold (1971) [2], Abeasis and del Fra (1985) [1], Pokrzywa (1986) [39], De Hoyos (1990) [9], Bongartz (1990)[7], Elmroth (1995) [16], and Boley (1998) [6]. Stratifications of various control applications are considered in [42, 20, 23, 24].

In our future work, we will extend the functionalities of StratiGraph in several directions. For example, interaction with Matlab is underway. In this way, we will extend StratiGraph with quantitative information on distances between different structures in a Jordan or Kronecker closure hierarchy.

We will also investigate other techniques that can be used for obtaining distance information. Some examples can be found in the papers of Boley (1990) [5], Elmroth and Kågström (1996) [18], Byers, He, and Mehrmann (1998) [8], and Gracia and De Hoyos (1999) [22, 21].

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