DIRECT EIGENVALUE REORDERING IN A PRODUCT OF MATRICES IN EXTENDED PERIODIC REAL SCHUR FORM*

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Abstract. A direct method for eigenvalue reordering in a product of a K-periodic matrix sequence in periodic or extended periodic real Schur form is presented and analyzed. Each reordering of two adjacent sequences of diagonal blocks is performed tentatively to guarantee backward stability and involves solving a K-periodic Sylvester equation (PSE) and constructing a K-periodic sequence of orthogonal transformation matrices. An error analysis of the direct reordering method is presented and results from computational experiments confirm the stability and accuracy of the method for well-conditioned as well as ill-conditioned problems. These include matrix sequences with fixed and time-varying dimensions, and sequences of small and large periodicity.

Key words. Product of K-periodic matrix sequence, extended periodic real Schur form, eigenvalue reordering, K-periodic Sylvester equation, periodic eigenvalue problem.

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1. Introduction.

The Schur form of a matrix is a fundamental tool in Numerical Linear Algebra: given a real square matrix A, there is a real orthogonal matrix Z, and a real quasi-triangular matrix T such that

$$Z^T A Z = T \equiv \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}. \tag{1.1}$$

The quasi-triangular T has 1×1 and 2×2 blocks on the main block diagonal corresponding to real and complex conjugate pairs of eigenvalues, respectively. If the 2×2 blocks are in standard form (see, e.g., [9]), the *real Schur form* T is essentially unique, except for the ordering of the eigenvalues, which can appear in any desired order along the block diagonal, and scaling of the transformation matrices. Moreover, the p first columns of Q span an *invariant subspace* of A corresponding to the p eigenvalues located in the upper-left part T_{11} [22]. The real Schur form is typically computed by means of a reduction to upper Hessenberg form followed by applying the QR-algorithm to the resulting Hessenberg matrix [9].

For a K-periodic (or K-cyclic) real matrix sequence, $A_0,\ A_1, \ldots, A_{K-1}$ with $A_K = A_0$, there exists a periodic counterpart of the real Schur form, called the *periodic real Schur form* (PRSF) [5, 11]: given the real matrix sequence $A_k \in R^{n \times n}$, for $k = 0, 1, \ldots, K-1$, there exists an orthogonal matrix sequence $Z_k \in R^{n \times n}$ such that the real sequence

$$Z_{k+1}^T A_k Z_k = T_k,\ k = 0, 1, \ldots, K-1, \tag{1.2}$$

with $Z_K = Z_0$, consists of $K-1$ upper triangular matrices and one upper quasi-triangular matrix. The products of conforming 1×1 and 2×2 diagonal blocks of the matrix sequence T_k give the eigenvalues of the matrix product $A_{K-1} \cdots A_1 A_0$.

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Similar to the standard case ($K = 1$), the periodic real Schur form is computed by means of a reduction to periodic Hessenberg form followed by applying a periodic QR-algorithm to the resulting sequence [5, 11]. The PRSF is an important tool in several applications, including solving periodic Sylvester-type and Riccati matrix equations [11, 20, 23, 26]. The quasi-triangular matrix in the PRSF can occur anywhere in the sequence, but is usually chosen to be T_0 or T_{K-1}. The *extended periodic real Schur form* (EPRSF) generalizes PRSF to the case when the dimensions of the matrices are time-variant [24]: given the real matrix sequence $A_k \in R^{n_{k+1} \times n_k}$, $k = 0, 1, \ldots, K-1$, with $n_K = n_0$, there exists an orthogonal matrix sequence $Z_k \in R^{n_k \times n_k}$, $k = 0, 1, \ldots, K-1$, such that the real sequence

$$Z_{k+1}^T A_k Z_k = T_k \equiv \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} \in R^{n_{k+1} \times n_k}, \tag{1.3}$$

for $k = 0, 1, \ldots, K-1$, with $Z_K = Z_0$, is block upper triangular and $T_{11}^{(k)} \in R^{\min_k(n_k) \times \min_k(n_k)}$, $T_{22}^{(k)} \in R^{(n_{k+1} - \min_k(n_k)) \times (n_k - \min_k(n_k))}$. Moreover, the subsequence $T_{11}^{(k)}$, $k = 0, 1, \ldots, K-1$, is in PRSF (1.2) with eigenvalues called the *core characteristic values* of the sequence A_k and the matrices in the subsequence $T_{22}^{(k)}$, $k = 0, 1, \ldots, K-1$, are upper trapezoidal. For EPRSF, the quasi-triangular matrix can occur at any position in the sequence T_k. However, to simplify the reduction to extended periodic Hessenberg form it is normally placed at position j, where $n_{j+1} = \min_k(n_k)$, i.e., in the matrix T_j which has the smallest row dimension in the sequence [24]. For T_j, $j \in [0, K-1]$, to have a trapezoidal block $T_{22}^{(j)}$, it must hold that $n_j, n_{j+1} > \min_k(n_k)$. The EPRSF is motivated by the increasing interest in *discrete-time periodic systems* of the form

$$\begin{array}{rcl} x_{k+1} & = & A_k x_k + B_k u_k \\ y_k & = & C_k x_k + D_k u_k, \end{array} \tag{1.4}$$

where the matrices $A_k \in R^{n_{k+1} \times n_k}$, $B_k \in R^{n_{k+1} \times m}$, $C_k \in R^{r \times n_k}$ and $D_k \in R^{r \times m}$ are periodic with periodicity $K \geq 1$. The *state transition matrix* of the system (1.4) is defined as the $n_j \times n_i$ matrix $\Phi_A(j, i) = A_{j-1} A_{j-2} \ldots A_i$, where $\Phi_A(i, i) = I_{n_i}$. The state transition matrix over one whole period $\Phi_A(j + K, j) \in R^{n_j \times n_j}$ is called the *monodromy matrix* of (1.4) at time j and its eigenvalues are called the *characteristic multipliers* at time j. All t nonzero and $(\min_k(n_k) - t)$ zero characteristic multipliers belong to the set of core characteristic values. One important issue is how to reorder the eigenvalues of the monodromy matrix without evaluating the corresponding product. Evaluating the product is both costly and may lead to a significant loss of accuracy [5], especially in the eigenvalues of small magnitude.

Eigenvalue reordering in the real Schur form was investigated in [2, 8, 7], and for the *generalized Schur form* of a regular matrix pencil $A - \lambda B$ in [14, 16]. Reordering of eigenvalues in PRSF and related problems have also been considered, see, e.g., [5] where the approach is based on applying Givens rotations on explicitly formed products of small (2×2, 3×3 or 4×4) matrix sequences, and [6] for a discussion on swapping 1×1 blocks by propagating orthogonal transformations through 2×2 sequences. In this paper, we present a direct swapping algorithm for doing eigenvalue reordering in a product of a K-periodic matrix sequence in EPRSF for $K \geq 2$ without evaluating any part of the matrix product. Our direct algorithm relies on orthogo-

nal transformations only and extends earlier work on direct eigenvalue reordering of matrices and matrix pencils to products of matrices.

The rest of this paper is organized as follows. In Section 2, we settle some important notation and definitions. In Section 3, we discuss reordering of two diagonal blocks (leaving the eigenvalues invariant) by periodic orthogonal transformations, and in Section 4, we present our direct periodic reordering algorithm. Next, we discuss the numerical solution of the associated periodic Sylvester equation (PSE) in Section 5. An error analysis of the direct periodic swapping algorithm is presented in Section 6. Some numerical examples are presented and discussed in Section 7, and finally, we give some concluding remarks in Section 8.

2. Notation and definitions. We introduce some notation to simplify the presentation that follows. Let I_n denote the identity matrix of order n. Let M^+ denote the pseudo-inverse (see, e.g., [9]) of a matrix M. Let $\sigma(M)$ and $\lambda(M)$ denote the sets of the singular values and the eigenvalues of the matrix M, respectively. Let $A \otimes B$ denote the Kronecker product of two matrices, defined as the matrix with its (i,j)-block element as $a_{ij}B$. Let $\mathrm{vec}(A)$ denote a vector representation of an $m \times n$ matrix A with the columns of A stacked on top of each other in the order $1, 2, \ldots, n$. Let $\|A\|_F$ denote the Frobenius matrix norm defined as $\|A\|_F = \sqrt{\mathrm{trace}(A^T A)}$. We define the *periodic addition operator* $\oplus$ such that $a \oplus b = (a+b) \bmod K$, where K denotes the periodicity. We use the product operator $\prod_{k=i}^{j} B_k$ to denote a product $B_i B_{i-1} \cdots B_{j+1} B_j$, with the convention that $\prod_{k=i}^{j} B_k = I$ for $i < j$.

Each K-periodic matrix sequence A_k is associated with a *matrix tuple* $\bar{A} = (A_{K-1}, A_{K-2}, \ldots, A_1, A_0)$ [4]. The vector tuple $\bar{u} = (u_{K-1}, u_{K-2}, \cdots, u_1, u_0)$, with $u_k \neq 0$, is called a *right eigenvector* of the tuple $\bar{A}$ corresponding to the eigenvalue λ if there exist scalars α_k, possibly complex, such that the relations

$$\begin{aligned} &A_k u_k = \alpha_k u_{k\oplus 1}, \ k = 0, 1, \ldots, K-1, \\ &\lambda := \textstyle\prod_{k=K-1}^{0} \alpha_k \end{aligned} \tag{2.1}$$

hold with $u_K = u_0$. A *left eigenvector* $\bar{v}$ of the tuple $\bar{A}$ corresponding to λ is defined similarly

$$\begin{aligned} &v_{k\oplus 1}^H A_k = \beta_k v_k^H, \ k = 0, 1, \ldots, K-1, \\ &\lambda := \textstyle\prod_{k=K-1}^{0} \beta_k, \end{aligned} \tag{2.2}$$

where $v_k \neq 0$, and β_k are (possibly complex) scalars for $k = 0, 1, \ldots, K-1$. If $\bar{u}$ and $\bar{v}$ are unit-norm representations of the right and left eigenvectors corresponding to the simple eigenvalue λ of $\bar{A}$, respectively, the numbers

$$\kappa_k \equiv |v_k^H u_k| > 0, \tag{2.3}$$

are *reciprocal condition numbers* of the eigenvalue (characteristic multiplier at time k) λ of $\Phi_A(K+k, k)$.

Without loss of generality, we assume that $p < \min_k (n_k)$ is specified such that no 2×2 block corresponding to a complex conjugate pair of eigenvalues is positioned at rows (and columns) p and $p+1$ of $\Phi_T(K, 0)$. Given such a p and with Z_k and T_k from (1.3), the leading p columns of each Z_k span an invariant subspace for $\Phi_T(K+k, k)$ for $k = 0, 1, \ldots, K-1$. As a whole, the space spanned by the first p columns of each matrix in the matrix tuple $\bar{Z}$ is called a *right periodic deflating subspace* of the tuple $\bar{A}$ corresponding to the p eigenvalues located in the upper-leftmost part of $\Phi_T(K, 0)$. In general, $\Phi_T(K, 0)_{ij}$ denotes the (i,j)-block of the matrix product $\Phi_T(K, 0)$.

3. Reordering diagonal blocks in a product of matrices in EPRSF by orthogonal transformations. Consider the K-periodic (or K-cyclic) matrix sequences $A_k \in R^{n_{k\oplus 1}\times n_k}$, $T_k \in R^{n_{k\oplus 1}\times n_k}$ and $Z_k \in R^{n_k\times n_k}$, $k = 0, 1, \ldots, K-1$, such that A_k is a general matrix, T_k is in EPRSF and Z_k is the corresponding orthogonal transformation, as in (1.3). Then the eigenvalues of the product $\Phi_T(K,0) = T_{K-1}T_{K-2}\ldots T_1T_0 \in R^{n_0\times n_0}$ are diagonal blocks of size 1×1 (real) and 2×2 (complex conjugate pairs) of $\Phi_T(K,0)$.

Assume that each $T_k,\ k = 0, 1, \ldots, K-1$ is partitioned as

$$T_k = \left[\begin{array}{c|ccc} T_{11}^{(k)} & \star & \star & \star \\ \hline 0 & T_{22}^{(k)} & \star & \star \\ 0 & 0 & T_{33}^{(k)} & \star \\ \hline 0 & 0 & 0 & T_{44}^{(k)} \end{array}\right], \tag{3.1}$$

where $T_{11}^{(k)} \in R^{p_1\times p_1}$, $T_{22}^{(k)} \in R^{p_2\times p_2}$, $T_{33}^{(k)} \in R^{p_3\times p_3}$ and $T_{44}^{(k)} \in R^{(n_{k\oplus 1}-p)\times(n_k-p)}$, $k = 0, 1, \ldots, K-1$ and $p = p_1 + p_2 + p_3$. Notice that $\Phi_T(K,0)$ can be partitioned conformally such that

$$\Phi_T(K,0)_{ii} = T_{ii}^{(K-1)}T_{ii}^{(K-2)}\ldots T_{ii}^{(1)}T_{ii}^{(0)},\ i = 1, \ldots, 4. \tag{3.2}$$

Assume for a moment that there exists a *K-cyclic orthogonal matrix sequence* $Q_k,\ k = 0, 1, \ldots, K-1$, such that we can form the *cyclic transformation*

$$Q_{k\oplus 1}^T \left[\begin{array}{cc} T_{22}^{(k)} & \star \\ 0 & T_{33}^{(k)} \end{array}\right] Q_k = \left[\begin{array}{cc} \hat{T}_{22}^{(k)} & \star \\ 0 & \hat{T}_{33}^{(k)} \end{array}\right] \tag{3.3}$$

and $\lambda(\Phi_{\hat{T}}(K,0)_{22}) = \lambda(\Phi_T(K,0)_{33})$, $\lambda(\Phi_{\hat{T}}(K,0)_{33}) = \lambda(\Phi_T(K,0)_{22})$. In other words, this cyclic transformation swaps the eigenvalues of the $(2,2)$-block and the $(3,3)$-block of the matrix product $\Phi_T(K,0)$. Apparently, each matrix in the sequence Q_k has order $p_2 + p_3$. Then the reordered EPRSF of the sequence A_k is the sequence $\hat{T}_k$, where

$$\begin{aligned} \hat{T}_k &= \underbrace{\left[\begin{array}{c|cc} I_{p_1} & 0 & 0 \\ \hline 0 & Q_{k\oplus 1}^T & 0 \\ 0 & 0 & I_{p_4} \end{array}\right]}_{\hat{Q}_{k\oplus 1}^T} \left[\begin{array}{c|ccc} T_{11}^{(k)} & \star & \star & \star \\ \hline 0 & T_{22}^{(k)} & \star & \star \\ 0 & 0 & T_{33}^{(k)} & \star \\ \hline 0 & 0 & 0 & T_{44}^{(k)} \end{array}\right] \underbrace{\left[\begin{array}{c|cc} I_{p_1} & 0 & 0 \\ \hline 0 & Q_k & 0 \\ 0 & 0 & I_{p_4} \end{array}\right]}_{\hat{Q}_k} \\ &= \hat{Q}_{k\oplus 1}^T T_k \hat{Q}_k = \hat{Q}_{k\oplus 1}^T Z_{k\oplus 1}^T A_k Z_k \hat{Q}_k = \hat{Z}_{k\oplus 1}^T A_k \hat{Z}_k, \end{aligned} \tag{3.4}$$

with the associated K-cyclic orthogonal sequence $\hat{Z}_k = Z_k\hat{Q}_k,\ k = 0, 1, \ldots, K-1$. The first $p_1 + p_3$ columns of $\hat{Z}_0$ span an orthonormal basis for the invariant subspace of $\Phi_A(K,0)$ associated with the $p_1 + p_3$ first eigenvalues in the upper left part of the product $\Phi_{\hat{T}}(K,0)$. In addition, the first p_1+p_3 columns of each transformation matrix $\hat{Z}_k$ in the tuple $(\hat{Z}_{K-1}, \hat{Z}_{K-2}, \ldots, \hat{Z}_1, \hat{Z}_0)$ span an orthonormal basis for the periodic deflating subspace of the tuple $\bar{A}$ associated with the same $p_1 + p_3$ first eigenvalues in $\Phi_{\hat{T}}(K,0)$.

4. A direct algorithm for periodic diagonal block reordering. Without loss of generality, we assume that T_k in (3.1) is square, i.e., the sequence T_k is in PRSF, and partitioned as

$$T_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix}, \ k = 0, 1, \ldots, K-1, \tag{4.1}$$

and that we want to swap the blocks $T_{11}^{(k)} \in R^{p_1 \times p_1}$ and $T_{22}^{(k)} \in R^{p_2 \times p_2}$. Throughout the paper we assume that $\Phi_T(K,0)_{11}$ and $\Phi_T(K,0)_{22}$ have no eigenvalues in common, otherwise, the diagonal blocks need not be swapped. Define the K-cyclic matrix sequence $\mathbf{X}_k$ as

$$\mathbf{X}_k \equiv \begin{bmatrix} I_{p_1} & X_k \\ 0 & I_{p_2} \end{bmatrix}, \tag{4.2}$$

where $X_k \in R^{p_1 \times p_2}$, $k = 0, 1, \ldots, K-1$. The key observation is that the cyclic transformation

$$\mathbf{X}_{k\oplus 1}^{-1} \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} \mathbf{X}_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} + T_{11}^{(k)} X_k - X_{k\oplus 1} T_{22}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} \tag{4.3}$$

block-diagonalizes T_k, $k = 0, 1, \ldots, K-1$, if and only if the sequence X_k satisfies the *periodic Sylvester equation* (PSE)

$$T_{11}^{(k)} X_k - X_{k\oplus 1} T_{22}^{(k)} = -T_{12}^{(k)}, \ k = 0, 1, \ldots, K-1. \tag{4.4}$$

Replacing I_{p_2} in $\mathbf{X}_0$ (4.2) by a $p_2 \times p_2$ zero block results in a spectral projector (e.g., see [22]) associated with the matrix product $\Phi_T(K,0)$ that projects onto the spectrum of $\Phi_T(K,0)_{11}$. We refer to the matrix X_0 as the *generator matrix* for the periodic reordering of the product $\Phi_T(K,0)$.

Define the permutation matrices

$$P_l = \begin{bmatrix} 0 & I_{p_2} \\ I_{p_1} & 0 \end{bmatrix}, \qquad P_r = \begin{bmatrix} 0 & I_{p_1} \\ I_{p_2} & 0 \end{bmatrix},$$

and observe that $P_l P_r = P_r P_l = I_{p_1+p_2}$. The similarity transformation

$$\begin{aligned}
&P_l \mathbf{X}_0^{-1} (T_{K-1} T_{K-2} \ldots T_1 T_0) \mathbf{X}_0 P_r \\
&= \underbrace{P_l \mathbf{X}_0^{-1}}_{S_0^{-1}} T_{K-1} \underbrace{\mathbf{X}_{K-1} P_r}_{S_{K-1}} \underbrace{P_l \mathbf{X}_{K-1}^{-1}}_{S_{K-1}^{-1}} T_{K-2} \underbrace{\mathbf{X}_{K-2} P_r}_{S_{K-2}} \cdots \\
&\ldots \underbrace{P_l \mathbf{X}_2^{-1}}_{S_2^{-1}} T_1 \underbrace{\mathbf{X}_1 P_r}_{S_1} \underbrace{P_l \mathbf{X}_1^{-1}}_{S_1^{-1}} T_0 \underbrace{\mathbf{X}_0 P_r}_{S_0} \\
&= S_0^{-1} T_{K-1} S_{K-1} S_{K-1}^{-1} T_{K-2} S_{K-2} \ldots S_2^{-1} T_1 S_1 S_1^{-1} T_0 S_0 \\
&= \begin{bmatrix} T_{22}^{(K-1)} & 0 \\ 0 & T_{11}^{(K-1)} \end{bmatrix} \cdots \begin{bmatrix} T_{22}^{(1)} & 0 \\ 0 & T_{11}^{(1)} \end{bmatrix} \begin{bmatrix} T_{22}^{(0)} & 0 \\ 0 & T_{11}^{(0)} \end{bmatrix},
\end{aligned}$$

performs the wanted swapping of the diagonal blocks by the non-orthogonal sequence

$$S_k = \mathbf{X}_k P_r = \begin{bmatrix} X_k & I_{p_1} \\ I_{p_2} & 0 \end{bmatrix}, \ k = 0, 1, \ldots, K-1.$$

Since the first p_2 columns of each S_k are linearly independent there exist orthogonal matrices Q_k of order $p_1 + p_2$ such that

$$D_k \equiv \begin{bmatrix} X_k \\ I_{p_2} \end{bmatrix} = Q_k \begin{bmatrix} R_k \\ 0 \end{bmatrix}, \tag{4.5}$$

where R_k of size $p_2 \times p_2$ is upper triangular and non-singular, $k = 0, 1, \ldots, K-1$. By partitioning Q_k conformally with S_k, we observe that

$$Q_k^T S_k = \begin{bmatrix} R_k & {Q_{11}^{(k)}}^T \\ 0 & {Q_{12}^{(k)}}^T \end{bmatrix}, \qquad S_k^{-1} Q_k = \begin{bmatrix} R_k^{-1} & -R_k^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T} \\ 0 & {Q_{12}^{(k)}}^{-T} \end{bmatrix}.$$

An orthonormal similarity transformation of $\Phi_T(K, 0)$ can now be written as

$$Q_0^T (T_{K-1} T_{K-2} \ldots T_1 T_0) Q_0 = Q_0^T T_{K-1} Q_{K-1} Q_{K-1}^T T_{K-2} Q_{K-2} \ldots Q_2^T T_1 Q_1 Q_1^T T_0 Q_0$$

$$= Q_0^T S_0 \begin{bmatrix} T_{11}^{(K-1)} & 0 \\ 0 & T_{22}^{(K-1)} \end{bmatrix} S_{K-1}^{-1} Q_{K-1} Q_{K-1}^T S_{K-1} \begin{bmatrix} T_{11}^{(K-2)} & 0 \\ 0 & T_{22}^{(K-2)} \end{bmatrix} S_{K-1}^{-1} Q_{K-2} \ldots$$

$$\ldots Q_2^T S_2 \begin{bmatrix} T_{11}^{(1)} & 0 \\ 0 & T_{22}^{(1)} \end{bmatrix} S_1^{-1} Q_1 Q_1^T S_1 \begin{bmatrix} T_{11}^{(0)} & 0 \\ 0 & T_{22}^{(0)} \end{bmatrix} S_0^{-1} Q_0 = \hat{T}_{K-1} \hat{T}_{K-2} \ldots \hat{T}_1 \hat{T}_0,$$

where

$$\hat{T}_k = \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ 0 & \hat{T}_{22}^{(k)} \end{bmatrix}$$

and

$$\begin{cases} \hat{T}_{11}^{(k)} &= R_{k \oplus 1} T_{22}^{(k)} R_k^{-1} \\ \hat{T}_{22}^{(k)} &= {Q_{12}^{(k \oplus 1)}}^T T_{11}^{(k)} {Q_{12}^{(k)}}^{-T} \\ \hat{T}_{12}^{(k)} &= -R_{k \oplus 1} T_{22}^{(k)} R_k^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T} + {Q_{11}^{(k \oplus 1)}}^T T_{11}^{(k)} {Q_{12}^{(k)}}^{-T} \end{cases}, \tag{4.6}$$

for $k = 0, 1, \ldots, K-1$. Thus, the orthogonal sequence Q_k from (4.5) performs the required reordering of the diagonal blocks. Observe that the sequences $\hat{T}_{11}^{(k)}$ and $\hat{T}_{22}^{(k)}$ in (4.6) may not be in PRSF and might have to be further transformed after periodic reordering by additional orthogonal transformations to get the sequence $\hat{T}_k$ in PRSF.

We summarize our direct algorithm for periodic eigenvalue reordering as follows:
Step 1 Solve for the sequence X_k, $k = 0, 1, \ldots, K-1$, in the PSE

$$T_{11}^{(k)} X_k - X_{k \oplus 1} T_{22}^{(k)} = -T_{12}^{(k)}, \ k = 0, 1, \ldots, K-1.$$

Step 2 Compute K orthogonal matrices Q_k such that

$$\begin{bmatrix} X_k \\ I_{p_2} \end{bmatrix} = Q_k \begin{bmatrix} R_k \\ 0 \end{bmatrix}, \ k = 0, 1, \ldots, K-1.$$

Step 3 Perform reordering by the cyclic transformations

$$\hat{T}_k = Q_{k\oplus 1}^T T_k Q_k,\ k = 0, 1, \ldots, K-1. \tag{4.7}$$

Step 4 Restore the subsequences $\hat{T}_{11}^{(k)}$ and $\hat{T}_{22}^{(k)}$ to periodic real Schur form using K-cyclic orthogonal transformations.

Step 4 is conducted by computing PRSFs of the two K-periodic subsequences $\hat{T}_{11}^{(k)}$ and $\hat{T}_{22}^{(k)}$. Care must be taken to assure that each of the two quasi-triangular matrices in the PRSFs appear in the same position of the $\hat{T}_k$ sequence, say $\hat{T}_i$.

In the presence of rounding errors, the most critical step in the reordering process is to solve the periodic Sylvester equation. In analogy to eigenvalue swapping in the real (generalized) Schur form, a small sep-function (defined in Equation (5.6) of the next section) may ruin backward stability and thus forces us to perform the swapping tentatively to guarantee backward stability [2, 14, 16]. See also Kressner [17] for a brief discussion on direct swapping methods for PRSF.

The direct algorithm extends directly to EPRSF by considering reordering of the core characteristic values (see Section 2) of the sequence T_k.

5. The periodic Sylvester equation. By basic algebraic manipulations the PSE in (4.4) can be reduced into K independent triangular continuous-time Sylvester equations (SYCT) of the form

$$F_k X_k - X_k G_k = C_k,\ k = 0, 1, \ldots, K-1, \tag{5.1}$$

where

$$F_k = \Phi_T(k,0)_{11}\Phi_T(K,k)_{11},\ G_k = \Phi_T(k,0)_{22}\Phi_T(K,k)_{22},\ k = 0, \ldots, K-1,$$

and C_k is a sum of combinations of blocks in the subsequences $T_{11}^{(i)}$, $T_{22}^{(i)}$ and $T_{12}^{(i)}$, $i = 0, 1, \ldots, K-1$. By applying the forward error bound for the standard Sylvester equation [2, 13] to the periodic case, we get the following norm-wise forward error bound for each matrix in the solution to the PSE (4.4), where $\tilde{X}_k$ is the computed solution, X_k the exact, ρ_k is a small scalar and ϵ_{mach} is the relative machine precision:

$$\frac{\|X_k - \tilde{X}_k\|}{\|X_k\|} \leq \frac{\rho_k \epsilon_{\text{mach}}(\|F_k\|_F + \|G_k\|_F)}{\text{sep}(F_k, G_k)}, \tag{5.2}$$

where

$$\text{sep}(F_k, G_k) = \inf_{\|X_k\|_F = 1} \|F_k X_k - X_k G_k\|_F = \sigma_{\min}(I_{p_2} \otimes F_k - G_k \otimes I_{p_1}). \tag{5.3}$$

It is clear that all F_k have the same eigenvalues independently of k (this also holds for G_k), but $\text{sep}(F_k, G_k)$ may differ for different values of k since matrix multiplication in general is not commutative. We conclude that the PSE (4.4) has a unique solution sequence $X_k,\ k = 0, 1, \ldots, K-1$, if and only if (5.1) has a unique solution for all $k = 0, 1, \ldots, K-1$, and it is well-known [3] that this holds when $\lambda(F_k) \neq \lambda(G_k)$ or equivalently $\text{sep}(F_k, G_k) > 0$, for all $k = 0, 1, \ldots, K-1$. See also Lemma 2.1 in [4] for a similar result.

Without going into details, the PSE in (4.4) can be solved in several ways. We can reduce it into K single continuous-time Sylvester equations by evaluating the products

F_k and G_k to compute X_k for all values of k. This may not be suitable for large periods or large dimensions from accuracy considerations. In analogy with solving the standard Sylvester equation (e.g., see [3]), we construct a matrix representation Z_{PSE} of the periodic Sylvester operator defined by the PSE (4.4) in terms of Kronecker products, where

$$Z_{\mathrm{PSE}} = \tag{5.4}$$
$$\begin{bmatrix} {T_{22}^{(K-1)}}^T \otimes I_{p_1} & & & & I_{p_2} \otimes T_{11}^{(K-1)} \\ I_{p_2} \otimes T_{11}^{(0)} & {T_{22}^{(0)}}^T \otimes I_{p_1} & & & \\ & \ddots & \ddots & & \\ & & & I_{p_2} \otimes T_{11}^{(K-2)} & {T_{22}^{(K-2)}}^T \otimes I_{p_1} \end{bmatrix}.$$

Only the nonzero blocks of Z_{PSE} are displayed explicitly in (5.4). Then we solve the resulting linear system of equations $Z_{\mathrm{PSE}} x = c$, with x and c as stacked vector representations of the matrix sequences X_k, for $k = 0, 1, \ldots, K-1$, and C_k, $k = K-1, 0, 1, \ldots, K-2$, respectively:

$$x = \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \\ \cdots \\ \mathrm{vec}(X_{K-1}) \end{bmatrix}, \qquad c = \begin{bmatrix} \mathrm{vec}(C_{K-1}) \\ \mathrm{vec}(C_0) \\ \cdots \\ \mathrm{vec}(C_{K-2}) \end{bmatrix}. \tag{5.5}$$

Observe that the dimension of Z_{PSE} grows linearly with K. To exploit the structure of the matrix Z_{PSE}, Gaussian elimination with partial pivoting (GEPP) should be used to the cost of $O(K(p_1^2 p_2 + p_1 p_2^2))$ flops, possibly combined with fixed precision iterative refinement for improved accuracy on badly scaled problems. By storing only the block main diagonal, the block subdiagonal and the rightmost block column vector, the storage requirement for Z_{PSE} can be kept at $3Kp_1^2p_2^2$. One could employ Gaussian elimination with complete pivoting (GECP) to solve this linear system (see, e.g., LAPACK's `DTGSYL` [16]), but that would make it difficult, if not impossible, to exploit the sparsity structure of the problem. The complete pivoting process causes fill-in elements, requires explicit storage of the whole matrix Z_{PSE} and increases the number of flops to $O((Kp_1p_2)^3)$.

Also in analogy with the standard Sylvester equation (e.g., see [12, 15] and (5.3)), the conditioning of the periodic Sylvester equation is related to the sep-function

$$\mathrm{sep}[\mathrm{PSE}] = \inf_{\|x\|_2=1} \|Z_{\mathrm{PSE}} x\|_2 = \|Z_{\mathrm{PSE}}^{-1}\|_2^{-1} = \sigma_{\min}(Z_{\mathrm{PSE}}) \tag{5.6}$$
$$= \inf_{(\sum_{k=0}^{K-1} \|X_k\|_F^2)^{1/2}=1} \Big(\sum_{k=0}^{K-1} \|T_{11}^{(k)} X_k - X_{k\oplus 1} T_{22}^{(k)}\|_F^2\Big)^{1/2}.$$

The quantity sep[PSE] can be estimated at the cost of solving a few PSEs by exploiting the estimation technique for the 1-norm of the inverse of a matrix [10, 12, 15, 16].

6. Error Analysis. In this section, we present an error analysis of the direct reordering method presented in Section 4, where we extend the analysis from [2, 14] to the periodic case. First we investigate the impact of the cyclic transformations (3.3) on the individual matrices in the sequence T_k (4.1), for $k = 0, 1, \ldots, K-1$. Secondly, we apply our findings to the diagonal blocks of the explicitly formed matrix product $\Phi_{\hat{T}}(K, 0)$.

6.1. Perturbation of individual matrices under periodic reordering. If Householder reflections are used to compute the orthogonal sequence $\tilde{Q}_k$, $k = 0, 1, \ldots, K-1$, each matrix $\tilde{Q}_k$ is orthogonal up to machine precision [27] and the stability of the direct reordering method is mainly affected by the conditioning and accuracy of the solution to the associated periodic Sylvester equation.

Without loss of generality, we assume that $p_1 = p_2 = 2$. Let $\tilde{X}_k$ be the computed solution sequence to the PSE (4.4), where $\tilde{X}_k = X_k + \Delta X_k$, X_k is the exact and unique solution sequence and ΔX_k is the corresponding error matrix for $k = 0, 1, \ldots, K-1$. We let

$$Y_k \equiv T_{11}^{(k)} \tilde{X}_k - \tilde{X}_{k\oplus 1} T_{22}^{(k)} + T_{12}^{(k)} = T_{11}^{(k)} \Delta X_k - \Delta X_{k\oplus 1} T_{22}^{(k)} \tag{6.1}$$

denote the residual sequence associated with the computed PSE solution sequence.

Under mild conditions (such as $\|D_k^+\|_2 \|\Delta X_k\|_F < 1$, where D_k is defined in (4.5)) the K QR-factorizations of $(\tilde{X}_k,\ I)^T$ can be written as

$$\begin{bmatrix} X_k + \Delta X_k \\ I \end{bmatrix} = D_k + \begin{bmatrix} \Delta X_k \\ 0 \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} \tilde{R}_k \\ 0 \end{bmatrix} = (Q_k + \Delta Q_k) \begin{bmatrix} R_k + \Delta R_k \\ 0 \end{bmatrix},$$

where ΔQ_k and ΔR_k are perturbations of the orthogonal matrices Q_k and the triangular matrices R_k, and $\tilde{Q}_k = Q_k + \Delta Q_k$ is orthogonal [21]. Here, $\|\Delta Q_k\|_F$ and $\|\Delta R_k\|_F$ are essentially bounded by $\|D_k^+\|_2 \|\Delta X_k\|_F$, $k = 0, 1, \ldots, K-1$ [21, 2]. We do not assume anything about the structure of these perturbation matrices.

Given the computed sequences $\tilde{X}_k$ and $\tilde{Q}_k$, the following theorem shows how the errors in these quantities propagate to the results of the direct method for reordering two adjacent sequences of diagonal blocks in the periodic Schur form.

THEOREM 6.1. *Let $\tilde{X}_k = X_k + \Delta X_k$ with $\Delta X_k \neq 0$ nonsingular, $\tilde{Q}_k$, and the residual sequence Y_k (6.1) be given for $k = 0, 1, \ldots, K-1$. By applying the computed sequence of transformations $\tilde{Q}_k$ from a periodic reordering of the (1,1) and (2,2) blocks of T_k (4.1) in a cyclic transformation, we get*

$$\tilde{T}_k \equiv \tilde{Q}_{k\oplus 1}^T \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} \tilde{Q}_k = \hat{T}_k + E_k, \tag{6.2}$$

where

$$\hat{T}_k = \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ 0 & \hat{T}_{22}^{(k)} \end{bmatrix}, \qquad E_k = \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{bmatrix}, \tag{6.3}$$

for $k = 0, 1, \ldots, K-1$. Then the error matrices E_k satisfy the following norm bounds up to first order perturbations:

$$\begin{aligned} \|E_{11}^{(k)}\|_2 \leq\ & \frac{\sigma_{\max}(X_{k\oplus 1})}{(1 + \sigma_{\max}^2(X_{k\oplus 1}))^{1/2}} \cdot \frac{1}{(1 + \sigma_{\min}^2(X_k))^{1/2}} \|Y_k\|_F \\ & + 2\|\hat{T}_{11}^{(k)}\|_2 (\|D_k^+\|_2 \|\Delta X_k\|_F + \|D_{k\oplus 1}^+\|_2 \|\Delta X_{k\oplus 1}\|_F), \end{aligned} \tag{6.4}$$

$$\|E_{21}^{(k)}\|_2 \leq \frac{1}{(1 + \sigma_{\min}^2(X_{k\oplus 1}))^{1/2}} \cdot \frac{1}{(1 + \sigma_{\min}^2(X_k))^{1/2}} \|Y_k\|_F, \tag{6.5}$$

$$\|E_{22}^{(k)}\|_2 \leq \frac{1}{(1 + \sigma_{\min}^2(X_{k\oplus 1}))^{1/2}} \cdot \frac{\sigma_{\max}(X_k)}{(1 + \sigma_{\max}^2(X_k))^{1/2}} \|Y_k\|_F. \tag{6.6}$$

Proof. Transform the sequence T_k with $\tilde{Q}_k$ in a cyclic transformation:

$$\tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k = (Q_{k\oplus 1} + \Delta Q_{k\oplus 1})^T T_k (Q_k + \Delta Q_k)$$

$$= \underbrace{Q_{k\oplus 1}^T T_k Q_k}_{\hat{T}_k} + \Delta Q_{k\oplus 1}^T T_k Q_k + Q_{k\oplus 1}^T T_k \Delta Q_k + \Delta Q_{k\oplus 1}^T T_k \Delta Q_k.$$

Let $Z_k = Q_k^T \Delta Q_k$. From $(Q_k + \Delta Q_k)^T (Q_k + \Delta Q_k) = I$ we have that $Q_k^T \Delta Q_k = -\Delta Q_k^T Q_k$ up to first order and by dropping the second order term, we get

$$\tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k = \hat{T}_k + \Delta Q_{k\oplus 1}^T Q_{k\oplus 1} \underbrace{Q_{k\oplus 1}^T T_k Q_k}_{\hat{T}_k} + \underbrace{Q_{k\oplus 1}^T T_k Q_k}_{\hat{T}_k} Q_k^T \Delta Q_k$$

$$= \hat{T}_k + \hat{T}_k Q_k^T \Delta Q_k - Q_{k\oplus 1}^T \Delta Q_{k\oplus 1} \hat{T}_k = \hat{T}_k + \hat{T}_k Z_k - Z_{k\oplus 1} \hat{T}_k,$$

for $k = 0, 1, \ldots, K-1$.

Let E_k denote the error matrix corresponding to the k^{th} cyclic transformation (4.7), i.e., $\tilde{T}_k = \hat{T}_k + E_k$. Partition Z_k, $k = 0, 1, \ldots, K-1$ conformally with $\hat{T}_k$ and observe that

$$\tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k = \hat{T}_k + E_k = \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ 0 & \hat{T}_{22}^{(k)} \end{bmatrix} + \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{bmatrix},$$

where

$$\begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{bmatrix} = \hat{T}_k Z_k - Z_{k\oplus 1} \hat{T}_k$$

$$= \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ 0 & \hat{T}_{22}^{(k)} \end{bmatrix} \begin{bmatrix} Z_{11}^{(k)} & Z_{12}^{(k)} \\ Z_{21}^{(k)} & Z_{22}^{(k)} \end{bmatrix} - \begin{bmatrix} Z_{11}^{(k\oplus 1)} & Z_{12}^{(k\oplus 1)} \\ Z_{21}^{(k\oplus 1)} & Z_{22}^{(k\oplus 1)} \end{bmatrix} \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ 0 & \hat{T}_{22}^{(k)} \end{bmatrix}$$

i.e.,

$$\left\{ \begin{array}{lll} E_{11}^{(k)} = & \hat{T}_{11}^{(k)} Z_{11}^{(k)} + \hat{T}_{12}^{(k)} Z_{21}^{(k)} - Z_{11}^{(k\oplus 1)} \hat{T}_{11}^{(k)} \\ E_{12}^{(k)} = & \hat{T}_{11}^{(k)} Z_{12}^{(k)} + \hat{T}_{12}^{(k)} Z_{22}^{(k)} - Z_{11}^{(k\oplus 1)} \hat{T}_{12}^{(k)} - Z_{12}^{(k\oplus 1)} \hat{T}_{22}^{(k)} \\ E_{21}^{(k)} = & \hat{T}_{22}^{(k)} Z_{21}^{(k)} - Z_{21}^{(k\oplus 1)} \hat{T}_{11}^{(k)} \\ E_{22}^{(k)} = & \hat{T}_{22}^{(k)} Z_{22}^{(k)} - Z_{22}^{(k\oplus 1)} \hat{T}_{22}^{(k)} - Z_{21}^{(k\oplus 1)} \hat{T}_{12}^{(k)}. \end{array} \right. \tag{6.7}$$

As we will show below, $E_{22}^{(k)}$ and $E_{11}^{(k)}$ perturb the eigenvalues of the matrix product $\Phi_A(K, 0)$ directly, but do not affect stability (the error in block $(2,1)$ in the matrix $\tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k$). $E_{21}^{(k)}$ is critical since it affects both the stability of the reordering and the eigenvalues. $E_{12}^{(k)}$ is of minor interest since it does not perturb the eigenvalues explicitly nor affects the stability. The task is now to derive norm bounds for the error matrix blocks $E_{11}^{(k)}$, $E_{21}^{(k)}$ and $E_{22}^{(k)}$.

By assuming that ΔX_k, $k = 0, 1, \ldots, K-1$, are non-singular and applying the analysis of the QR-factorization from [2] to each of our K independent QR-factorizations, we get

$$Z_{11}^{(k)} = {Q_{11}^{(k)}}^T \Delta X_k R_k^{-1} - \Delta R_k R_k^{-1}, \tag{6.8}$$

$$Z_{21}^{(k)} = {Q_{12}^{(k)}}^T \Delta X_k R_k^{-1}, \tag{6.9}$$

$$Z_{22}^{(k)} = -{Q_{12}^{(k)}}^T \Delta X_k R_k^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T}. \tag{6.10}$$

Using Equations (6.8), (6.9), (6.10), and (4.6), the error matrix blocks $E_{11}^{(k)}$, $E_{21}^{(k)}$ and $E_{22}^{(k)}$ in (6.7) boil down to

$$\begin{aligned}
E_{11}^{(k)} &= \hat{T}_{11}^{(k)} Z_{11}^{(k)} + \hat{T}_{12}^{(k)} Z_{21}^{(k)} - Z_{11}^{(k\oplus 1)} \hat{T}_{11}^{(k)} \\
&= {Q_{11}^{(k\oplus 1)}}^T (T_{11}^{(k)} \Delta X_k - \Delta X_{k\oplus 1} T_{22}^{(k)}) R_k^{-1} - \hat{T}_{11}^{(k)} \Delta R_k R_k^{-1} + \Delta R_{k\oplus 1} R_{k\oplus 1}^{-1} \hat{T}_{11}^{(k)}, \\
E_{21}^{(k)} &= \hat{T}_{22}^{(k)} Z_{21}^{(k)} - Z_{21}^{(k\oplus 1)} \hat{T}_{11}^{(k)} \\
&= {Q_{12}^{(k\oplus 1)}}^T (T_{11}^{(k)} \Delta X_k - \Delta X_{k\oplus 1} T_{22}^{(k)}) R_k^{-1}, \\
E_{22}^{(k)} &= \hat{T}_{22}^{(k)} Z_{22}^{(k)} - Z_{22}^{k\oplus 1} \hat{T}_{22}^{(k)} - Z_{21}^{(k\oplus 1)} \hat{T}_{12}^{(k)} \\
&= -{Q_{12}^{(k\oplus 1)}}^T (T_{11}^{(k)} \Delta X_k - \Delta X_{k\oplus 1} T_{22}^{(k)}) R_k^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T},
\end{aligned}$$

as first order results. Since $Y_k = T_{11}^{(k)} \Delta X_k - \Delta X_{k\oplus 1} T_{22}^{(k)}$, the $E_{ij}^{(k)}$ blocks above can be expressed as

$$\begin{cases}
E_{11}^{(k)} = {Q_{11}^{(k\oplus 1)}}^T Y_k R_k^{-1} - \hat{T}_{11}^{(k)} \Delta R_k R_k^{-1} + \Delta R_{k\oplus 1} R_{k\oplus 1}^{-1} \hat{T}_{11}^{(k)} \\
E_{21}^{(k)} = {Q_{12}^{(k\oplus 1)}}^T Y_k R_k^{-1} \\
E_{22}^{(k)} = -{Q_{12}^{(k\oplus 1)}}^T Y_k R_k^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T}.
\end{cases} \tag{6.11}$$

We see that $E_{22}^{(k)}$, $E_{21}^{(k)}$ and $E_{11}^{(k)}$ are essentially related to the K residual matrices Y_k of the associated periodic Sylvester equation and the blocks R_k, $Q_{11}^{(k)}$ and $Q_{12}^{(k)}$ from the K QR-factorizations.

From (4.5) we have that

$$Q_{21}^{(k)} = R_k^{-1}$$

and

$$R_k^T R_k = I + X_k^T X_k,$$

which gives

$$\sigma^2(R_k) = \lambda(R_k^T R_k) = \lambda(I + X_k^T X_k) = 1 + \lambda(X_k^T X_k) = 1 + \sigma^2(X_k).$$

By the above argument we get

$$\|Q_{21}^{(k)}\|_2 = \|R_k^{-1}\|_2 = \frac{1}{\sigma_{\min}(R_k)} = \frac{1}{(1 + \sigma_{\min}^2(X_k))^{1/2}}.$$

Further, from [21] we have

$$\|\Delta R_k R_k^{-1}\|_F \leq 2\|D_k^+\|_2 \|\Delta X_k\|_F,$$

and by the CS-decomposition of Q (see, e.g., [9, 22]) we get the following norm relations

$$\|Q_{21}^{(k)}\|_2 = \|Q_{12}^{(k)}\|_2, \ \|Q_{11}^{(k)}\|_2 = \|Q_{22}^{(k)}\|_2.$$

Now by combining these facts with (6.11) and applying the product and triangle inequalities for norms, we obtain the bounds of the theorem. □

Remark 1: For $K = 1$ and by the inequality $(1+\sigma^2_{\min}(X_k))^{-1/2} \geq (1+\sigma^2_{\max}(X_k))^{-1/2}$, the norm bounds of Theorem 6.1 can be further bounded from above to achieve

$$\|E_{11}\|_2 \leq \frac{\sigma_{\max}(X)}{(1+\sigma^2_{\min}(X))}\|Y\|_F + 4\|\hat{T}_{11}\|_2\|D^+\|_2\|\Delta X\|_F, \tag{6.12}$$

$$\|E_{21}\|_2 \leq \frac{1}{(1+\sigma^2_{\min}(X))}\|Y\|_F, \tag{6.13}$$

$$\|E_{22}\|_2 \leq \frac{\sigma_{\max}(X)}{(1+\sigma^2_{\min}(X))}\|Y\|_F, \tag{6.14}$$

which are the norm bounds from the main theorem of [2] on the perturbation of the eigenvalues under standard eigenvalue reordering in the real Schur form.

Remark 2: The second term of Equation (6.4) can be combined with Equations (5.1) and (5.2), and by using that $\|D_k^+\|_2 = \sigma_{\min}(R_k)^{-1} = (1+\sigma^2_{\min}(X_k))^{-1/2}$, we get

$$\begin{aligned} & 2\|\hat{T}_{11}^{(k)}\|_2(\|D_k^+\|_2\|\Delta X_k\|_F + \|D_{k\oplus 1}^+\|_2\|\Delta X_{k\oplus 1}\|_F) \leq \\ \leq\ & 2\|\hat{T}_{11}^{(k)}\|_2\rho\epsilon_{\text{mach}}\Big(\frac{\|X_k\|_F(\|F_k\|_F + \|G_k\|_F)}{(1+\sigma^2_{\min}(X_k))^{1/2}\text{sep}(F_k,G_k)} + \\ +\ & \frac{\|X_{k\oplus 1}\|_F(\|F_{k\oplus 1}\|_F + \|G_{k\oplus 1}\|_F)}{(1+\sigma^2_{\min}(X_{k\oplus 1}))^{1/2}\text{sep}(F_{k\oplus 1},G_{k\oplus 1})}\Big), \end{aligned} \tag{6.15}$$

where $\rho = \max(\rho_k, \rho_{k\oplus 1})$. However, the upper bound (6.15) is mostly of theoretical interest since it calls for explicit calculation of F_k and G_k and $\text{sep}(F_k, G_k)$.

Remark 3: Numerical experiments show that iterative refinement may improve on the computed solution X_k, especially for badly scaled problems, but may not improve on the residual sequence Y_k. See also [2] for a similar observation.

6.2. Perturbation of matrix products under periodic reordering. In this section, we investigate how the errors in the invidual matrices after a periodic reordering of two adjacent sequences of diagonal blocks in T_k propagate into the matrix product $\Phi_T(K,0) = T_{K-1}T_{K-2}\dots T_1T_0$.

We present a general result in the following theorem.

THEOREM 6.2. *Let T_k be a matrix sequence in PRSF with periodicity K and partitioned as*

$$T_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix}.$$

Let the sequence $\tilde{Q}_k$, $k = 0, 1, \dots, K-1$, be the computed orthogonal cyclic transformation matrices defining the periodic eigenvalue reordering of the product $\Phi_T(K,0)$ as in (6.2). In addition, let the sequences $\tilde{T}_k$, $\hat{T}_k$, and E_k be defined as in (6.2)–(6.3) of Theorem 6.1. Then, we have

$$\Phi_{\tilde{T}}(K,0) = \prod_{k=K-1}^{0} \tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k = \Phi_{\hat{T}}(K,0) + \mathbf{E}, \tag{6.16}$$

where $\Phi_{\hat{T}}(K,0) = Q_0^T \Phi_T(K,0) Q_0$ *is the exact product of the reordered matrices and* $\mathbf{E}$ *is the corresponding error matrix. Assuming that* $\mathbf{E}$ *is partitioned conformally with* T_k*, we have the bounds*

$$
\left\{
\begin{array}{rcl}
\|\mathbf{E}_{11}\|_2 & \leq & \sum_{k=0}^{K-1}((\prod_{j=K-1}^{k+1}\|\hat{T}_{11}^{(j)}\|_2)\|E_{11}^{(k)}\|_2 \\
 & & + (\sum_{j=K-1}^{k+1}\|\varphi_1^{(k,j)}\|_2)\|E_{21}^{(k)}\|_2)(\prod_{j=k-1}^{0}\|\hat{T}_{11}^{(j)}\|_2) \\
\|\mathbf{E}_{21}\|_2 & \leq & \sum_{k=0}^{K-1}(\prod_{j=K-1}^{k+1}\|\hat{T}_{22}^{(j)}\|_2)\|E_{21}^{(k)}\|_2(\prod_{j=k-1}^{0}\|\hat{T}_{11}^{(j)}\|) \\
\|\mathbf{E}_{22}\|_2 & \leq & \sum_{k=0}^{K-1}(\prod_{j=K-1}^{k+1}\|\hat{T}_{22}^{(j)}\|_2)(\|E_{21}^{(k)}\|_2\sum_{j=k-1}^{0}\|\varphi_2^{(k,j)}\|_2 \\
 & & + \|E_{22}^{(k)}\|_2(\prod_{j=k-1}^{0}\|\hat{T}_{22}^{(j)}\|_2))
\end{array}
\right. , \tag{6.17}
$$

where

$$
\|\varphi_1^{(k,j)}\|_2 \leq \|\hat{T}_{12}^{(j)}\|_2 \prod_{l=K-1}^{j+1}\|\hat{T}_{11}^{(l)}\|_2 \prod_{l=j-1}^{k+1}\|\hat{T}_{22}^{(l)})\|_2 \tag{6.18}
$$

$$
\|\varphi_2^{(k,j)}\|_2 \leq \|\hat{T}_{12}^{(j)}\|_2 \prod_{l=i-1}^{j+1}\|\hat{T}_{11}^{(l)}\|_2 \prod_{l=j-1}^{0}\|\hat{T}_{22}^{(l)}\|_2 \tag{6.19}
$$

up to first order perturbations.

Proof. Up to first order perturbations, we have

$$
\begin{array}{l}
\Phi_{\tilde{T}}(K,0) = \prod_{k=K-1}^{0}\tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k \\
= \Phi_{\hat{T}}(K,0) + \sum_{k=0}^{K-1}\Phi_{\hat{T}}(K,k+1)E_k\Phi_{\hat{T}}(k,0) = \Phi_{\hat{T}}(K,0) + \mathbf{E}.
\end{array} \tag{6.20}
$$

The error matrix $\mathbf{E}$ can be expressed in block partitioned form:

$$
\begin{array}{rcl}
\mathbf{E} & = & \sum_{k=0}^{K-1}\Phi_{\hat{T}}(K,k+1)E_k\Phi_{\hat{T}}(k,0) \\
 & = & \sum_{k=0}^{K-1}\left[\begin{array}{cc} \Phi_{\hat{T}}(K,k+1)_{11} & \sum_{j=K-1}^{k+1}\varphi_1^{(k,j)} \\ 0 & \Phi_{\hat{T}}(K,k+1)_{22} \end{array}\right] . \left[\begin{array}{cc} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{array}\right] \\
 & & . \left[\begin{array}{cc} \Phi_{\hat{T}}(k,0)_{11} & \sum_{j=k-1}^{0}\varphi_2^{(k,j)} \\ 0 & \Phi_{\hat{T}}(k,0)_{22} \end{array}\right] = \sum_{i=0}^{K-1}\left[\begin{array}{cc} \mathbf{E}_{11}^{(k)} & \mathbf{E}_{12}^{(k)} \\ \mathbf{E}_{21}^{(k)} & \mathbf{E}_{22}^{(k)} \end{array}\right],
\end{array} \tag{6.21}
$$

where

$$
\begin{array}{rcl}
\mathbf{E}_{11}^{(k)} & = & (\Phi_{\hat{T}}(K,k+1)_{11}E_{11}^{(k)} + \sum_{j=K-1}^{k+1}\varphi_1^{(k,j)}E_{21}^{(k)})\Phi_{\hat{T}}(k,0)_{11} \\
\mathbf{E}_{12}^{(k)} & = & (\Phi_{\hat{T}}(K,k+1)_{11}E_{11}^{(k)} + \sum_{j=K-1}^{k+1}\varphi_1^{(k,j)}E_{21}^{(k)})\sum_{j=k-1}^{0}\varphi_2^{(k,j)} + \\
 & + & (\Phi_{\hat{T}}(K,k+1)_{11}E_{12}^{(k)} + \sum_{j=K-1}^{k+1}\varphi_1^{(k,j)}E_{22}^{(k)})\Phi_{\hat{T}}(k,0)_{22} \\
\mathbf{E}_{21}^{(k)} & = & \Phi_{\hat{T}}(K,k+1)_{22}E_{21}^{(k)}\Phi_{\hat{T}}(k,0)_{11} \\
\mathbf{E}_{22}^{(k)} & = & \Phi_{\hat{T}}(K,k+1)_{22}(E_{21}^{(k)}\sum_{j=k-1}^{0}\varphi_2^{(k,j)} + E_{22}^{(k)}\Phi_{\hat{T}}(k,0)_{22})
\end{array}
$$

and

$$\varphi_1^{(k,j)} = \Phi_{\hat{T}}(K, j+1)_{11} \hat{T}_{12}^{(j)} \Phi_{\hat{T}}(j, k+1)_{22}$$
$$\varphi_2^{(k,j)} = \Phi_{\hat{T}}(k, j+1)_{11} \hat{T}_{12}^{(j)} \Phi_{\hat{T}}(j, 0)_{22}.$$

By applying the triangle inequality and the submultiplicativity of norms to these results, the bounds in the theorem follow. □

For illustration, we display the explicit results of Theorem 6.2 for two simple cases in the following Corollary.

COROLLARY 6.3. *Under the assumptions of Theorem 6.2, and the periodicity $K = 2$, norm bounds for blocks of the error matrix $\mathbf{E}$ (6.16) can up to first order perturbations be expressed as*

$$\begin{aligned}
\|\mathbf{E}_{11}\|_2 &\leq \|\hat{T}_{11}^{(1)}\|_2 \|E_{11}^{(0)}\|_2 + \|\hat{T}_{12}^{(1)}\|_2 \|E_{21}^{(0)}\|_2 + \|\hat{T}_{11}^{(0)}\|_2 \|E_{11}^{(1)}\|_2 \\
\|\mathbf{E}_{21}\|_2 &\leq \|\hat{T}_{22}^{(1)}\|_2 \|E_{21}^{(0)}\|_2 + \|\hat{T}_{11}^{(0)}\|_2 \|E_{21}^{(1)}\|_2 \\
\|\mathbf{E}_{22}\|_2 &\leq \|\hat{T}_{22}^{(1)}\|_2 \|E_{22}^{(0)}\|_2 + \|\hat{T}_{12}^{(0)}\|_2 \|E_{21}^{(1)}\|_2 + \|\hat{T}_{22}^{(0)}\|_2 \|E_{22}^{(1)}\|_2
\end{aligned}.$$

For periodicity $K = 3$, we have the bounds

$$\begin{aligned}
\|\mathbf{E}_{11}\|_2 \leq\ & \|\hat{T}_{11}^{(2)}\|_2 \|\hat{T}_{11}^{(1)}\|_2 \|E_{11}^{(0)}\|_2 + (\|\hat{T}_{11}^{(2)}\|_2 \|\hat{T}_{12}^{(1)}\|_2 + \|\hat{T}_{12}^{(2)}\|_2 \|\hat{T}_{22}^{(1)}\|_2) \|E_{21}^{(0)}\|_2 + \\
& + \|\hat{T}_{11}^{(2)}\|_2 \|\hat{T}_{11}^{(0)}\|_2 \|E_{11}^{(1)}\|_2 + \|\hat{T}_{12}^{(2)}\|_2 \|\hat{T}_{11}^{(0)}\|_2 \|E_{21}^{(1)}\|_2 + \|\hat{T}_{11}^{(1)}\|_2 \|\hat{T}_{11}^{(0)}\|_2 \|E_{11}^{(2)}\|_2 \\
\|\mathbf{E}_{21}\|_2 \leq\ & \|\hat{T}_{22}^{(2)}\|_2 \|\hat{T}_{22}^{(1)}\|_2 \|E_{21}^{(0)}\|_2 + \|\hat{T}_{22}^{(2)}\|_2 \|\hat{T}_{11}^{(0)}\|_2 \|E_{21}^{(1)}\|_2 + \|\hat{T}_{11}^{(1)}\|_2 \|\hat{T}_{11}^{(0)}\|_2 \|E_{21}^{(2)}\|_2 \\
\|\mathbf{E}_{22}\|_2 \leq\ & \|\hat{T}_{22}^{(2)}\|_2 \|\hat{T}_{22}^{(1)}\|_2 \|E_{22}^{(0)}\|_2 + \|\hat{T}_{22}^{(2)}\|_2 \|\hat{T}_{12}^{(0)}\|_2 \|E_{21}^{(1)}\|_2 + \|\hat{T}_{22}^{(2)}\|_2 \|\hat{T}_{22}^{(0)}\|_2 \|E_{22}^{(1)}\|_2 + \\
& + (\|\hat{T}_{11}^{(1)}\|_2 \|\hat{T}_{12}^{(0)}\|_2 + \|\hat{T}_{12}^{(1)}\|_2 \|\hat{T}_{22}^{(0)}\|_2) \|E_{21}^{(2)}\|_2 + \|\hat{T}_{22}^{(1)}\|_2 \|\hat{T}_{22}^{(0)}\|_2 \|E_{22}^{(2)}\|_2
\end{aligned}$$

up to first order perturbations.

We remark that the analysis in Theorem 6.2 and Corollary 6.3 assumes that the involved matrix products and sums are computed exactly. For a rounding error analysis regarding matrix products and sums, see, e.g., [13].

Theorems 6.1 and 6.2 can be combined to produce computable bounds for the perturbations of the diagonal blocks of $\Phi_{\tilde{T}}(K, 0)$ under periodic eigenvalue reordering. We can also apply known perturbation results for the standard eigenvalue problem [22] and the periodic eigenvalue problem [18, 4] to the submatrix products $\Phi_{\tilde{T}}(K, 0)_{11}$ and $\Phi_{\tilde{T}}(K, 0)_{22}$. This is a matter of further investigation.

7. Computational Experiments. We demonstrate the stability and reliability of the direct reordering method by considering some numerical examples. The test examples span from well-conditioned to ill-conditioned problems, including matrix sequences with fixed and time-varying dimensions, and sequences of small and large periodicity. In the following, we present results for a representative selection of problems, where, except for one example, two complex conjugate eigenvalue pairs of a periodic real sequence A_k are reordered ($p_1 = p_2 = 2$). The associated PSEs of our direct periodic reordering method are solved by applying Gaussian elimination with partial pivoting to $Z_{\mathrm{PSE}} x = c$ and utilizing the structure of Z_{PSE} in (5.4). All experiments are carried out in double precision ($\epsilon_{\mathrm{mach}} \approx 2.2 \times 10^{-16}$) on an UltraSparc II (450 Mhz) workstation.

Examples 1–2 below are constructed as follows. We first specify K, n_k, $k = 0, 1, \ldots, K-1$, and $\min_k(n_k)$ eigenvalues or $K \cdot \min_k(n_k)$ diagonal and $\min_k(n_k) - 1$ subdiagonal elements. Then a random sequence T_k as in (1.3) is generated with 1×1 and 2×2 diagonal blocks corresponding to specified eigenvalues or diagonal, sub- and superdiagonal entries. Finally, orthogonal matrices Z_k, $k = 0, 1, \ldots, K-1$, are constructed from QR-factorizing K uniformly distributed random $n_k \times n_k$ matrices, which are applied in a K-cyclic orthogonal transformation of T_k to get A_k. Optionally, the sequence A_k is scaled so that all $||A_k||_F$ are of the same size (within a factor 10). Examples 3 and 4 illustrate reordering of two periodic sequences already in PRSF. The last example, Example 5, is from a real application.

In Table 7.1, we display the periodicity K, problem dimensions n_k for $k = 0, 1, \ldots, K-1$, the computed value of sep[PSE], and a reciprocal condition number s for the eigenvalues of $\Phi_T(K, 0)_{11}$

$$s = 1/\sqrt{1 + \|X_0\|_F^2},$$

where X_0 is the generator matrix for the periodic reordering of $\Phi_T(K, 0)$ (see Section 4). The last two quantities signal the conditioning of the problems considered.

Results from periodic reordering using our direct method are presented in Table 7.2. We display the maximum relative change of the eigenvalues under the periodic reordering

$$e_\lambda = \max_k \frac{|\lambda_k - \tilde{\lambda}_k|}{|\lambda_k|}, \ \lambda_k \in \lambda(\Phi_T(K, 0)).$$

In addition, we display five residual quantities for the computed results. These include two stability tests used in our method, namely a *weak stability test*

$$R_{\text{weak}} = \max_k \|\tilde{Q}_{11}^{(k)} - X_k \tilde{Q}_{21}^{(k)}\|_F,$$

and a *strong stability test*

$$R_{\text{strong}} = \max_k(\|T_k - \tilde{Q}_{k\oplus 1}\tilde{T}_k\tilde{Q}_k^T\|_F, \|\tilde{T}_k - \tilde{Q}_{k\oplus 1}^T T_k \tilde{Q}_k\|_F),$$

which is the maximum residual norm associated with the cyclic transformations $\tilde{Q}_k$ used in the reordering. Tolerances for these tests can optionally be specified. The last three are the maximum residual norms of the (extended) periodic Schur decomposition (1.3) before and after reordering, computed as

$$R_{\text{eprsf}} = \max_k(\|A_k - Z_{k\oplus 1}T_k Z_k^T\|_F, \|T_k - Z_{k\oplus 1}^T A_k Z_k\|_F),$$

and

$$R_{\text{reord}} = \max_k(\|A_k - \tilde{Z}_{k\oplus 1}\tilde{T}_k\tilde{Z}_k^T\|_F, \|\tilde{T}_k - \tilde{Z}_{k\oplus 1}^T A_k \tilde{Z}_k\|_F),$$

and a relative orthogonality check over the whole period K after periodic reordering

$$R_{\text{orth}} = \frac{\max_k(\|I_{n_k} - \tilde{Z}_k^T\tilde{Z}_k\|_F, \|I_{n_k} - \tilde{Z}_k^T\tilde{Z}_k\|_F)}{\epsilon_{\text{mach}}}.$$

TABLE 7.1
Problem characteristics for the examples considered. 4a and 4b refer to Example 4 with period 2 and 100, respectively.

Example	K	n_k	sep[PSE]	s
1	3	4+k	6.9E-01	7.2E-01
2	120	4	4.7E-03	5.5E-01
3	10	2	9.9E+00	1.0E+00
4a	2	4	4.5E-15	1.1E-14
4b	100	4	1.3E-16	1.3E-16
5	2	4	6.2E+03	6.6E-01

For these three residual norms, the K-cyclic transformations Z_k and $\tilde{Z}_k$ correspond to Z_k and $\hat{Z}_k$ in (3.4), respectively.

The computed eigenvalues before and after the periodic reordering are presented to full machine accuracy under each example.

Example 1. We consider a time-varying sequence with $K = 3$ and $n_k = 4 + k,\ k = 0, 1, 2$, and eigenvalues $1.0 \pm 2.0i, -7.0 \pm 0.5i$. The computed eigenvalues of the matrix product $\Phi_T(K, 0) = T_2 T_1 T_0$ are

$$\begin{aligned} \lambda_1 &= 1.000000000000000 \pm 2.000000000000000i \\ \lambda_2 &= -7.000000000000001 \pm 5.000000000000001i. \end{aligned}$$

The spectrum is well separated. After the periodic reordering of the blocks we obtained $\tilde{\lambda}_1 = \lambda_2$ and $\tilde{\lambda}_2 = \lambda_1$ to full accuracy.

Example 2 – satellite control [25]. We consider reordering in a 4×4 periodic matrix sequence that describes a control system of a satellite on orbit around the earth. The periodicity is $K = 120$. The eigenvalues of the sequence are

$$\begin{aligned} \lambda_1 &= 0.9941836588706161 \pm 0.1076979685723037i \\ \lambda_2 &= 0.7625695885261465 \pm 0.6469061930874623i. \end{aligned}$$

The reordered eigenvalues are

$$\begin{aligned} \tilde{\lambda}_1 &= 0.7625695885261450 \pm 0.6469061930874582i \\ \tilde{\lambda}_2 &= 0.9941836588706161 \pm 0.1076979685723021i. \end{aligned}$$

This application example shows that periodic reordering works fine for well-conditioned problems with large periods as well.

Example 3. We consider reordering a sequence with $K = 10$, $p_1 = p_2 = 1$, and the computed sequence in PRSF is

$$T_k = \begin{bmatrix} 10^1 & t_{12}^{(k)} \\ 0 & 10^{-1} \end{bmatrix},\ k = 0, 1, \ldots, K-1.$$

The computed eigenvalues of the product $\Phi_T(K, 0)$ are

$$\begin{aligned} \lambda_1 &= 9.999999999999987 \times 10^9 \\ \lambda_2 &= 1.000000000000013 \times 10^{-10}. \end{aligned}$$

After the periodic reordering we obtain

$$\begin{aligned} \tilde{\lambda}_1 &= 1.000000000000015 \times 10^{-10} \\ \tilde{\lambda}_2 &= 9.999999999999989 \times 10^9. \end{aligned}$$

Reordering of 1×1 blocks in PRSF can be carried out by propagating a Givens rotation through the matrix product (in practice, a perfect shift periodic QR-step), but this process is not forward stable. For this example, the rotation approach does not deliver one single correct digit in the reordered eigenvalues, whereas the direct reordering method delivers an acceptable error in the eigenvalues.

Example 4. We consider a sequence already in PRSF with $K = 2$ and $n_k = 4,\ k = 0, 1$, and eigenvalues $0.2 \pm (1.2 + 10^{-14})i,\ 0.2 \pm 1.2i$. The computed eigenvalues of the matrix $\Phi_T(K, 0) = T_1 T_0$ are

$$\begin{aligned} \lambda_1 &= 0.200000000000000 \pm 1.200000000000001i \\ \lambda_2 &= 0.200000000000000 \pm 1.200000000000000i. \end{aligned}$$

The spectrum is not well separated. After the periodic reordering we obtained

$$\begin{aligned} \tilde{\lambda}_1 &= 0.200000000000000 \pm 1.200000000000000i \\ \tilde{\lambda}_2 &= 0.200000000000000 \pm 1.200000000000001i, \end{aligned}$$

so the periodic reordering was perfect even though the problem has very close eigenvalues. Indeed, we obtain reordered eigenvalues to full machine accuracy for periods up to 100.

Example 5. First, we consider a problem already in PRSF with large separation and $K = 2,\ n_k = 4,\ k = 0, 1$, and the eigenvalues $\epsilon_{\text{mach}}^{1/2} \pm \epsilon_{\text{mach}}^{1/2},\ \epsilon_{\text{mach}}^{-1/2} \pm \epsilon_{\text{mach}}^{-1/2}$. Moreover, the involved matrices have almost the same Frobenius norm ($\approx 1.8 \times 10^4$) but the matrices in the subsequences $T_{11}^{(k)}$ and $T_{22}^{(k)}$ have very different norms: $\|T_{11}^{(1)}\|_F \approx 1.4 \times 10^4,\ \|T_{11}^{(2)}\|_F \approx 1.4 \times 10^4,\ \|T_{22}^{(1)}\|_F \approx 7.0 \times 10^{-12},\ \|T_{22}^{(2)}\|_F \approx 8.6 \times 10^3$. The computed eigenvalues of the product $\Phi_T(K, 0)$ are

$$\begin{aligned} \lambda_1 &= 6.710886400000000 \times 10^7 \pm 6.710886400000003 \times 10^7 i \\ \lambda_2 &= 1.490116119384766 \times 10^{-8} \pm 1.490116119384766 \times 10^{-8} i. \end{aligned}$$

After the periodic reordering without diagonal scaling we obtain

$$\begin{aligned} \tilde{\lambda}_1 &= 1.168840447839719 \times 10^{-8} \pm 9.309493732240201 \times 10^{-9} i \\ \tilde{\lambda}_2 &= 6.710886400000001 \times 10^7 \pm 6.710886400000000 \times 10^7 i. \end{aligned}$$

The problem is well-conditioned in the sense of sep[PSE], the norm of the generator matrix (see s in Table 7.2) and the reordering passes the stability tests, but since the eigenvalues differ almost 16 orders of magnitude the relative error in the smallest eigenvalues become very large due to the finite precision arithmetic.

Next, we consider the same problem as above, but now we perform diagonal scaling $T_2 T_1 = T_2 D_2 D_2^{-1} T_1$ before periodic reordering such that the blocks $T_{22}^{(1)}$ and $T_{22}^{(2)}$ have about the same norm. After the periodic reordering with diagonal scaling we obtain

$$\begin{aligned} \tilde{\lambda}_1 &= 1.490116120748016 \times 10^{-8} \pm 1.490116125160257 \times 10^{-8} i \\ \tilde{\lambda}_2 &= 6.710886400000000 \times 10^7 \pm 6.710886400000001 \times 10^7 i. \end{aligned}$$

which is quite an improvement (8 orders of magnitude) compared to the results without scaling. Not surprisingly, periodic reordering is sensitive to large differences in the norms within the subsequences $T_{11}^{(k)}$ and $T_{22}^{(k)}$.

TABLE 7.2
Computational results for periodic reordering. 4a and 4b refer to Example 4 with period $K = 2$ and 100, respectively. 5a and 5b refer to Example 5 without scaling and with scaling.

Example	e_λ	R_{weak}	R_{strong}	R_{eprsf}	R_{reord}	R_{orth}
1	4.6E-16	2.2E-16	1.6E-15	4.7E-15	5.6E-15	1.3E+01
2	1.6E-15	2.9E-16	1.8E-15	9.0E-15	9.8E-15	2.0E+01
3	1.4E-15	1.9E-16	8.4E-15	7.3E-15	1.0E-14	4.1E+00
4a	3.6E-16	2.5E-16	1.4E-15	0	1.2E-15	2.1E+00
4b	3.7E-16	2.3E-16	3.2E-18	0	1.9E-15	3.6E+00
5a	2.2E-01	1.2E-16	6.6E-12	0	5.8E-12	3.3E+00
5b	2.0E-09	2.3E-16	4.3E-12	0	5.6E-12	3.3E+00

8. Some concluding remarks. In this paper, we have presented a direct method for eigenvalue reordering in the extended periodic real Schur form of a K-periodic matrix sequence. The basic building blocks in the direct reordering is the numerical solution of an associated periodic Sylvester equation and the construction of K orthogonal matrices that perform the required reordering by cyclic transformations. The presented error analysis shows that errors in the transformations are closely related to the accuracy and conditioning of the solution to the associated periodic Sylvester equation as well as the norms of the involved matrices in the periodic sequence. Therefore, we perform the reordering tentatively so that backward stability can be guaranteed.

Future research will focus on computing periodic eigenspaces with specified eigenvalues and associated error bounds based on condition estimation (see, e.g., [16]), as well as producing library-standard (LAPACK [1], SLICOT [19]) software for the eigenvalue reordering algorithm presented in this paper.

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