1, 2, 3 & Higher Dimensions

Fred Gustavson
Umea Univ. || class
May 10, 2007

Popular Explanation

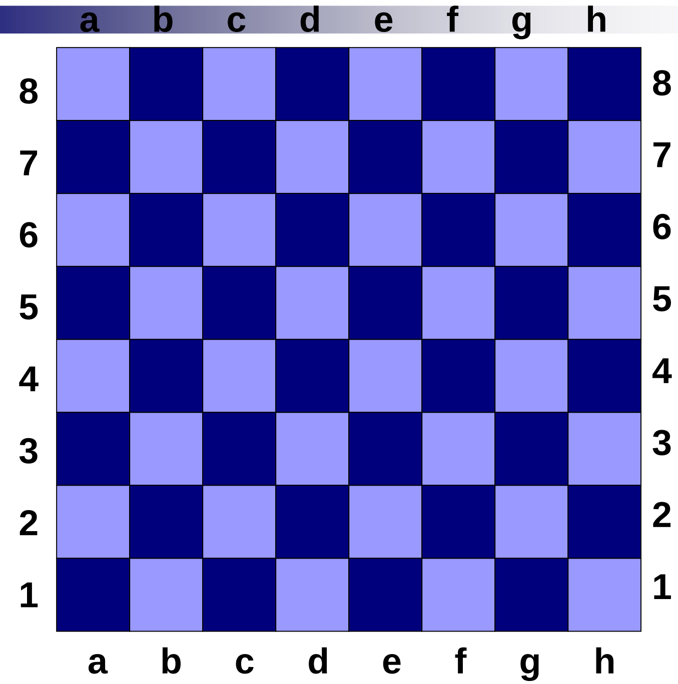
- Line has one dimension: length
- Surface; e.g., a piece of paper has two dimensions: length and width
- Space: e.g., a box has three dimensions: length, width and height
- Simple, clear and inadequate

Problems

- Line is okay
- Plane is okay if it is a rectangle; what about circles and ovals?
  - diameter is one dimensional; ellipses have variable diameters; yet these are 2-D
- Solid such as box is okay; what about a sphere?
  - one radius; yet it is called 3-D

Vague Definitions are Inadequate

- Study 2-D before going further
- Chess board
- City Maps



More on Chess

- Can play without board
- Need to visualize moves
- Label board horizontally and vertically

More on Maps

- Need to be able to identify your location
- Again a rectangle of squares labeled like a Chess board is in common use
- Tourist living in a hotel in Umea
  - finds his square
  - can easily walk to neighboring squares

Key Concept is a Neighborhood

- Does a labeling satisfy the neighborhood property of closeness?
- It will turn out that this notion can be made mathematically correct
- Hence, we will be able to define dimension in a satisfactory manner

Other labeling's

- Try natural Numbers: 1, 2, 3, …
- Examples on a Chess Board follow
- Notice: some neighboring squares are widely separated with this single labeling
- Same thing occurs for city maps
- Is this true for all single labeling's?

Five different labels follow

- CM or column major
- RM or row major
- Morton Z or recursive
- Integer to rational number mapping
- Two labels showing satisfaction of the neighborhood property

	a	b	c	d	e	f	g	h	
8	1	9	17	25	33	41	49	57	8
7	2	10	18	26	34	42	50	58	7
6	3	11	19	27	35	43	51	59	6
5	4	12	20	28	36	44	52	60	5
4	5	13	21	29	37	45	53	61	4
3	6	14	22	30	38	46	54	62	3
2	7	15	23	31	39	47	55	63	2
1	8	16	24	32	40	48	56	64	1
	a	b	c	d	e	f	g	h	

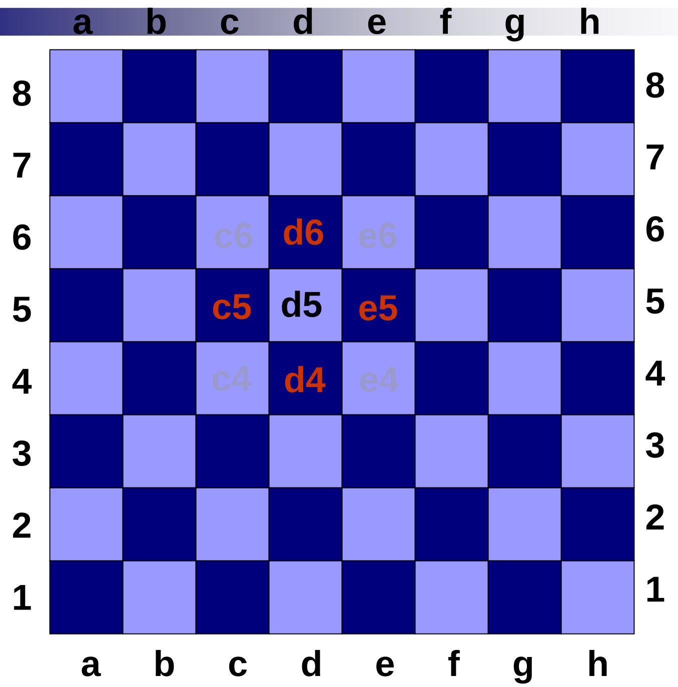
	a	b	c	d	e	f	g	h	
8	1	2	3	4	5	6	7	8	8
7	9	10	11	12	13	14	15	16	7
6	17	18	19	20	21	22	23	24	6
5	25	26	27	28	29	30	31	32	5
4	33	34	35	36	37	38	39	40	4
3	41	42	43	44	45	46	47	48	3
2	49	50	51	52	53	54	55	56	2
1	57	58	59	60	61	62	63	64	1
	a	b	c	d	e	f	g	h	

	a	b	c	d	e	f	g	h	
8	1	3	9	11	33	35	41	43	8
7	2	4	10	12	34	36	42	44	7
6	5	7	13	15	37	39	45	47	6
5	6	8	14	16	38	40	46	48	5
4	17	19	25	27	49	51	57	59	4
3	18	20	26	28	50	52	58	60	3
2	21	23	29	31	53	55	61	63	2
1	22	24	30	32	54	56	62	64	1
	a	b	c	d	e	f	g	h	

	a	b	c	d	e	f	g	h	
8	36	37	49	50	58	59	63	64	8
7	22	35	38	48	51	57	60	62	7
6	21	23	34	39	47	52	56	61	6
5	11	20	24	33	40	46	53	55	5
4	10	12	19	25	32	41	45	54	4
3	4	9	13	18	26	31	42	44	3
2	3	5	8	14	17	27	30	43	2
1	1	2	6	7	15	16	28	29	1
	a	b	c	d	e	f	g	h	

A metric for a Neighborhood

- Use a one norm: let p = (u,v) and q = (x,y) be two points
- Norm (p,q) = sum |u – v| + |x – y|



Cases where Natural Numbers suffice

- Years
- Temperature
- Milestones on a road

Mathematical Essence of Dimension

- Indexing with single numbers, or simple enumeration is applicable only to those cases where the objects have the character of a sequence
- Simple, single indexing must obey the neighborhood property. These objects are therefore labeled one dimensional

Two Dimensions

- Maps, Chessboards, etc. cannot be labeled by a simple sequential order
- Reason: the neighborhood property is violated
- However, two simple sequences suffice

2-D Labeling

- Rectangle: use Cartesian coordinates; x,y
- Circle: use polar coordinates; r,θ
- Surface of a torus: use two diameters
- Surface of a sphere: latitude and longitude
- Daily temperature in Umea: time and temperature

3-D Labeling

- Need three simple sequences
- Box: use Cartesian coordinates
- Solid Sphere: use spherical coordinate; r, θ, φ
- 3-D Chess

Dimension Number of a Domain

- Dimension: Number of numbers (symbols) to suitably characterize the elements of the domain
- Number of the numbers (symbols) give the dimension of the domain
  - line is 1-D, circle is 2-D, solid sphere is 3-D

Nature of Dimension

- Erroneous Notion: Rectangle has more points than a line; solid has more points than a rectangle
- Problem was corrected: All domains have the same number of points
- A problem remained: Is it possible to label a domain with two different labelings that both obey the neighborhood principle (higher to lower)
  - example: 2-D to 1-D

Theorem: Not possible

- LEJ Brouwer stated and proved this result in 1913.
- Some of Brouwer's methods were anticipated by Poincare