Umeå University
Department of Computing Science
TDBB08 – Logic with Applications
Examination: April 17, 1998

Name and ID number: ___________________________________

Signature:____

1. All answers must be written in English.
2. An English-X/X-English dictionary may be used, with X a natural language of your choice. No other help materials are allowed.
3. Solutions to all problems must be written on the special examination paper, which is provided. Write your name, ID number, and the problem number on each solution sheet. Collate the sheets in numerical order of the problems. Please write on only one side of the paper.
4. The examination has a total of 900 points. Your point total will be divided by 10, and then your average points from the obligatory exercises will be added on, to obtain a final score between 0 and 100, upon which your grade will be based.
5. In the table below, place an X in the position for any problem for which you have attempted a solution, and which you wish to have graded. **It is extremely important that you fill in this table properly**. For any box which is left blank, the associated question will not be graded, and you will instead be awarded 10% of the points for that question. Your decision to leave a box blank is definitive, so be very careful. For example, If you leave box 1(b) blank, your answer to that question will not be graded, even if it is completely correct. Similarly, if you place an X in box 1(b), but provide no answer whatsoever to that question, you will not receive 10% of the points for that question. It is strongly recommended that you use a pencil, in case you change your mind!

Prob.	1	2	3	4	5	6	7	8	9	10	11	12	13
(a)													
(b)		XXX	XXX	XXX	XXX	XXX	XXX		XXX	XXX	XXX	XXX	XXX
(c)	XXX	XXX	XXX	XXX	XXX	XXX	XXX		XXX	XXX	XXX	XXX	XXX

Notes:

- In case a problem has only one part, regard that part as being labelled with “(a).”

- Parts of problems which are labelled with (i), (ii), *etc.*, are not selectable individually for grading or 10% credit. The only options are to do the entire problem, or else take 10% of the points for that problem.

1. Let

$$\varphi_1 := ((A \vee B) \rightarrow ((C \vee D) \rightarrow (E \wedge F)))$$

(a) (35 points) Convert φ_1 to an equivalent formula in DNF (disjunctive normal form).
(b) (35 points) Convert φ_1 to an equivalent formula in CNF (conjunctive normal form).

2. Let

$$\Phi_2 := \{ D, \neg(A \wedge B), (B \vee C), \neg(C \wedge D \wedge E) \}$$
$$\varphi_2 := \neg E \vee \neg A$$

Using the method of semantic tableaux, either prove or disprove that $\Phi_2 \models \varphi_2$ is the case. Use the method and notation of the course notes, and not that of the textbook.

3. (70 points) Let

$\Phi_3 :=$
$\{ A \vee \neg C,\ \neg A \vee D,\ \neg A \vee \neg C,\ \neg B \vee C \vee E,\ A \vee C \vee \neg D,\ A \vee D \vee \neg E,$
$\neg A \vee B \vee C,\ B \vee C \vee D \vee E,\ C \vee \neg D \vee \neg E \}$

Using resolution, prove that $\Phi_3 \models \perp$ is the case. Display your solution in the form of a proof graph.

4. (70 points) Let

$$\Phi_4 := \{ B \vee \neg D,\ E \vee \neg C,\ D, \neg E,\ A \vee \neg B \vee C \vee \neg D,\ \neg A \vee C \}$$

Using input resolution, prove that $\Phi_4 \models \perp$ is the case. Display your solution in the form of a proof graph.

5. (70 points) Let

$$\Phi_5 := \{ ((\neg C) \rightarrow (A \vee B)), \neg(B \vee C) \}$$
$$\varphi_5 := A \wedge C$$

Using the Gentzen system G', either prove or disprove that $\Phi_5 \models \varphi_5$ is the case. The proof rules for this system are attached as an appendix to this examination.
Note: Do not normalize (*e.g.,* convert to CNF or negate goals) before beginning the solution. Begin with a sequent which contains **exactly** the formulas in $\Phi_5 \cup \{\varphi_5\}$. (Put them on the left-hand side or right-hand side of the initial sequent, as appropriate.)

6. (70 points) Given is the following formula.

$$(\forall x)(\exists w)(S(y,w) \wedge (\forall y)(P(x,y) \rightarrow Q(x,y)) \wedge (\forall x)(\forall w)(R(x,y) \wedge R(y,w)))$$

Do the following.

(i) Draw the parse tree for this formula, using the conventions given in the course notes.
(ii) Characterize each instance of each variable occurring in a term as free or bound. (Indicate these answers on your parse tree.)
(iii) Rename the variables in this formula, giving an equivalent formula with the property that no variable occurs in more than one quantifier, and no variable is both bound and free. In your solution, only bound variables may be renamed; free variables may not be renamed.

7. (70 points) Let

φ_7 :=
$$(\forall x)(\exists w)(R(x,w) \wedge ((\forall w)P(x,w) \rightarrow Q(x))) \vee (\forall x)(\forall w)(\exists y)(\neg(R(x,y) \wedge R(y,w)))$$

Apply the normalization algorithm to φ_7. Show at least the following intermediate formulas, all equivalent to φ_7.

(i) A formula in prenex normal form, with no $\rightarrow$ or $\leftrightarrow$ connectives. Quantifiers should be brought out in a form optimal for Skolemization.
(ii) A Skolemized version of your answer to (i).
(iii) A set of clauses with no common variables, derived from your answer to (ii).

In Problems 8-10, letters at the end of the alphabet (u, v, w, x, y, z) represent variables, and letters at the beginning of the alphabet (a, b, c) represent constants.

8. For each of the following pairs of atoms, either find a most general unifier, or else show that none exists. For this problem, do **not** rename variables before or during the unification process. Rather, assume that all occurrences of a given variable, even within the different terms, represent the same value.

(a: 20 points) $\{ P(f(x), y, g(z)),\ P(y, w, g(f(w))) \}$
(b: 20 points) $\{ Q(f(g(x)), g(f(y)), h(a,x)),\ Q(f(y), g(z), h(w,w)) \}$
(c: 20 points) $\{R(f(x), g(b), f(x)),\ R(y, z, f(y)) \}$

9. (70 points) Let

Φ_9 := { $(\forall x)(\forall y)(\neg P(x,y) \vee R(x,f(x)) \vee R(f(x),x))$,
$(\forall x)(\forall y)(P(x,g(y)) \vee \neg Q(g(x),y))$, $(\forall x)(\forall y)(P(x,a) \vee Q(g(x),y))$ }
$\varphi_9 := (\exists x)(\exists y)R(f(x),f(y))$

Prove that $\Phi_9 \models \varphi_9$ by converting this problem to an equivalent refutation problem, and then showing that the resulting set of clauses is unsatisfiable using resolution. Express your solution in the form of a proof graph, and include the substitutions which were used in the unifications.

10. (70 points) Let
Φ_{10} := { $(\forall x)(R(x) \vee P(x))$, $(\forall x)(\neg R(b) \vee P(x))$, $R(b) \vee (a=b)$ }
$\varphi_{10} := P(a) \wedge P(b)$

Prove that $\Phi_{10} \models \varphi_{10}$ by converting this problem to an equivalent refutation problem, and then showing that the resulting set of clauses is unsatisfiable using resolution with paramodulation. Express your solution in the form of a proof graph, and include the substitutions which were used in the unifications, as well as the replacements used in any application of paramodulation. Again, remember that it is permissible to rename variables within distinct clauses.

11. (70 points) Explain why the Hilbert proof system is not well-suited to use in automated theorem proving. You may restrict your answer to propositional logic. However, your answer must be reasonably precise, and must define any concepts (other than the basic components of a propositional logic) which it uses.

12. (70 points) In the context of sequent proof systems, the following two terms were introduced as properties of proof rules:
    (i) non-weakening;
    (ii) simplifying.

Define each, and explain why they are important in the context of a sequent-based proof system.

13. (70 points) The resolution proof process is sound and complete for refutation of sets of clauses, but not for the direct proof of a clause from a set of clauses. Illustrate this by providing a set Φ_{13} of (propositional) clauses, and a single clause φ_{13}, such that φ_{13} cannot be proven directly from Φ_{13}, yet this inference can be established indirectly via a refutation. Your answer must satisfy the following conditions:

(i) Φ_{13} is a satisfiable set of clauses.

(ii) φ_{13} is not a tautology.

(iii) If Ψ is a proper subset of Φ_{13} (*i.e.*, $\Psi \subseteq \Phi_{13}$ yet $\Psi \neq \Phi_{13}$), then $\Psi \cup \{\neg\varphi_{13}\}$ is satisfiable.

The fourteen proof rules of the Gentzen system G′ in common format are given below. In each case, the α_i's are to be bound to wff's, and the Γ_i's are to be bound to sequences of wff's.

$\dfrac{\Gamma_1, \alpha_1, \alpha_2, \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \wedge \alpha_2), \Gamma_2 \Rightarrow \Gamma_3}$	Left-$\wedge$
$\dfrac{\Gamma_1 \Rightarrow \Gamma_2, \alpha_1, \Gamma_3 \qquad \Gamma_1 \Rightarrow \Gamma_2, \alpha_2, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \wedge \alpha_2), \Gamma_3}$	Right-$\wedge$
$\dfrac{\Gamma_1, \alpha_1, \Gamma_2 \Rightarrow \Gamma_3 \qquad \Gamma_1, \alpha_2, \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \vee \alpha_2), \Gamma_2, \Rightarrow \Gamma_3}$	Left-$\vee$
$\dfrac{\Gamma_1 \Rightarrow \Gamma_2, \alpha_1, \alpha_2, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \vee \alpha_2), \Gamma_3}$	Right-$\vee$
$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow \alpha_1, \Gamma_3 \qquad \alpha_2, \Gamma_1, \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \rightarrow \alpha_2), \Gamma_2 \Rightarrow \Gamma_3}$	Left-$\rightarrow$
$\dfrac{\alpha_1, \Gamma_1 \Rightarrow \alpha_2, \Gamma_2, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \rightarrow \alpha_2), \Gamma_3}$	Right-$\rightarrow$
$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow \alpha, \Gamma_3}{\Gamma_1, \neg\alpha, \Gamma_2 \Rightarrow \Gamma_3}$	Left-$\neg$
$\dfrac{\alpha, \Gamma_1 \Rightarrow \Gamma_2, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, \neg\alpha, \Gamma_3}$	Right-$\neg$
$\dfrac{\Gamma_1, (\alpha_1 \rightarrow \alpha_2), (\alpha_2 \rightarrow \alpha_1), \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \leftrightarrow \alpha_2), \Gamma_2 \Rightarrow \Gamma_3}$	Left-$\leftrightarrow$
$\dfrac{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \rightarrow \alpha_2), \Gamma_3 \qquad \Gamma_1 \Rightarrow \Gamma_2, (\alpha_2 \rightarrow \alpha_1), \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \leftrightarrow \alpha_2), \Gamma_3}$	Right-$\leftrightarrow$
$\dfrac{\Gamma_1, \alpha_1, \alpha_2, .., \alpha_n, \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \wedge \alpha_2 \wedge .. \wedge \alpha_n), \Gamma_2 \Rightarrow \Gamma_3}$	Left-$\wedge$*
$\dfrac{\Gamma_1 \Rightarrow \Gamma_2, \alpha_1, \Gamma_3 \qquad \Gamma_1 \Rightarrow \Gamma_2, \alpha_2, \Gamma_3 \quad .. \quad \Gamma_1 \Rightarrow \Gamma_2, \alpha_n, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \wedge \alpha_2 \wedge .. \wedge \alpha_n), \Gamma_3}$	Right-$\wedge$*
$\dfrac{\Gamma_1, \alpha_1, \Gamma_2 \Rightarrow \Gamma_3 \qquad \Gamma_1, \alpha_2, \Gamma_2 \Rightarrow \Gamma_3 \quad .. \quad \Gamma_1, \alpha_n, \Gamma_2 \Rightarrow \Gamma_3}{\Gamma_1, (\alpha_1 \vee \alpha_2 \vee .. \vee \alpha_n), \Gamma_2, \Rightarrow \Gamma_3}$	Left-$\vee$*
$\dfrac{\Gamma_1 \Rightarrow \Gamma_2, \alpha_1, \alpha_2, .., \alpha_n, \Gamma_3}{\Gamma_1 \Rightarrow \Gamma_2, (\alpha_1 \vee \alpha_2 \vee .. \vee \alpha_n), \Gamma_3}$	Right-$\vee$*