A Short Tour of Tractability Hunting

With a special focus on Parameterized Complexity

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On Guest Lectures

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Course only starting: lot of fundamentals remain. Yet I want to say something aspirational

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As such I will give a *short tour* of some complexity theory concepts ahead of their time

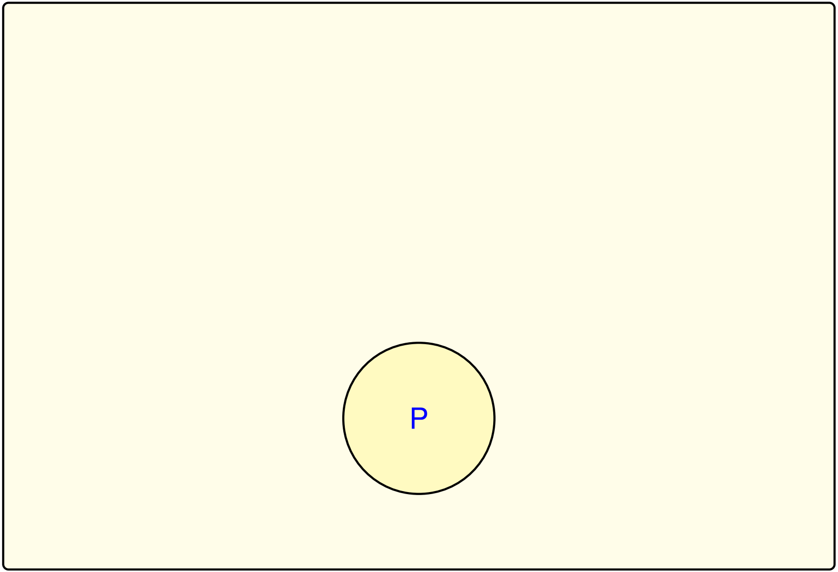
I will assume:

- Propositional logic
- A basic idea of what complexity, P and NP, *means*
- Some minor math and $\mathcal{O}$-notation

Will hopefully explain the rest

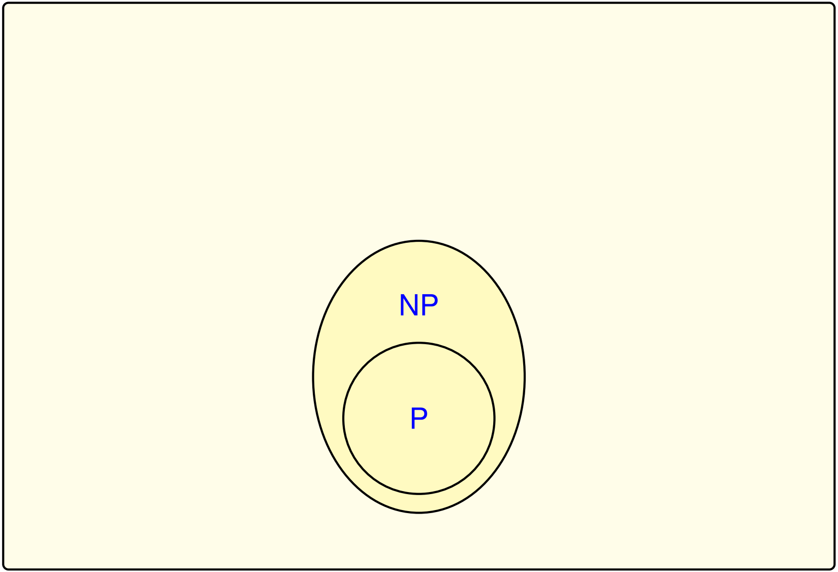
Part I: Complexity Background and Tractability

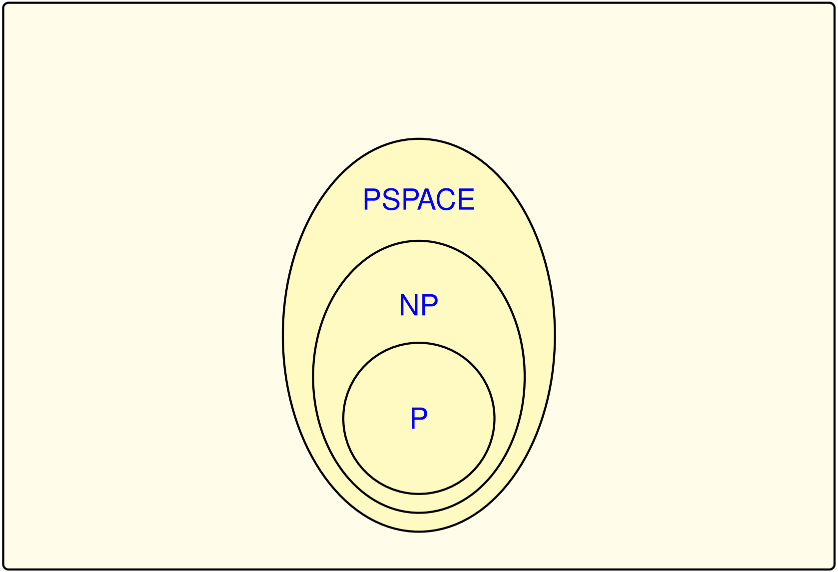
Classical complexity



big-oh, poly, circuit eval, moore's

Classical complexity

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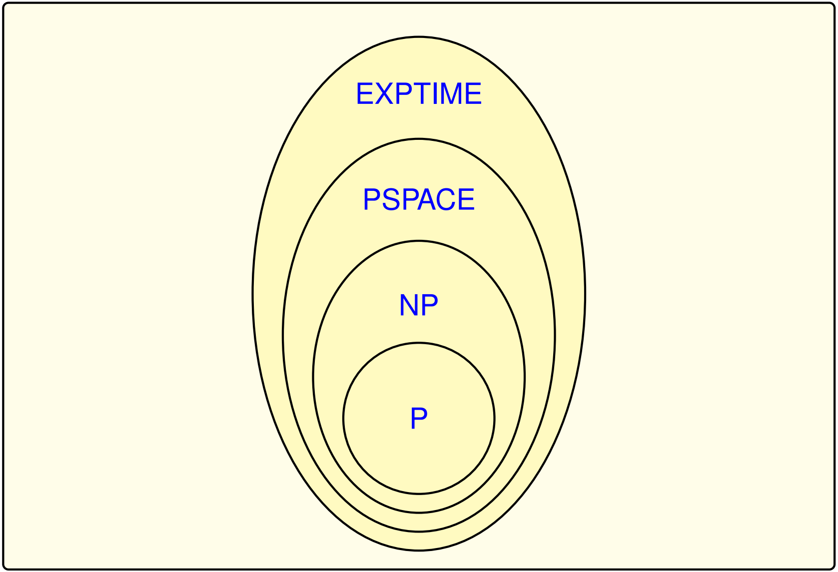
NP
P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Tractability = P?

Objection 1: A nicely behaved exponential is sometimes better than an ugly polynomial.

$$\begin{array}{ccc} 2^n & \textit{vs.} & n^{10} \\ 2^{50} & < & 50^{10} \end{array}$$

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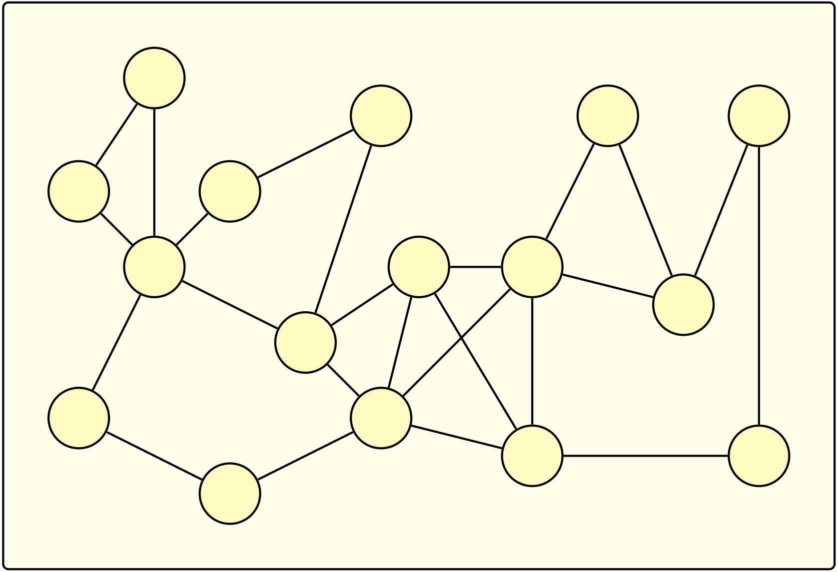
Objection 2: We simply cannot afford to consider all NP-hard problems intractable.

Still, we cannot just consider NP *tractable*: Cook was right!

VERTEX COVER

Input: A graph $G = (V, E)$, a constant $k \in N$.
Output: "Yes" if and only if there exists some $V' \subseteq V$, $|V'| \leq k$ such that every edge in E touches a vertex in V'.

Vertex Cover: Example



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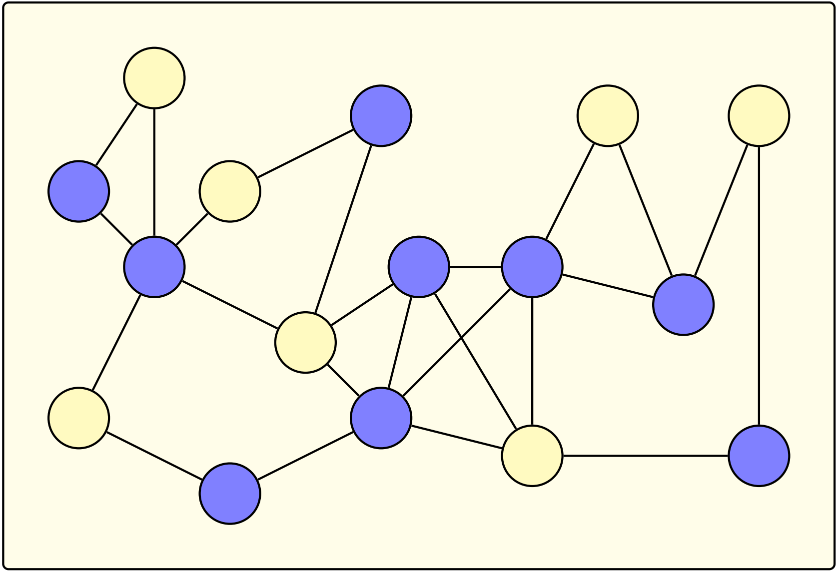
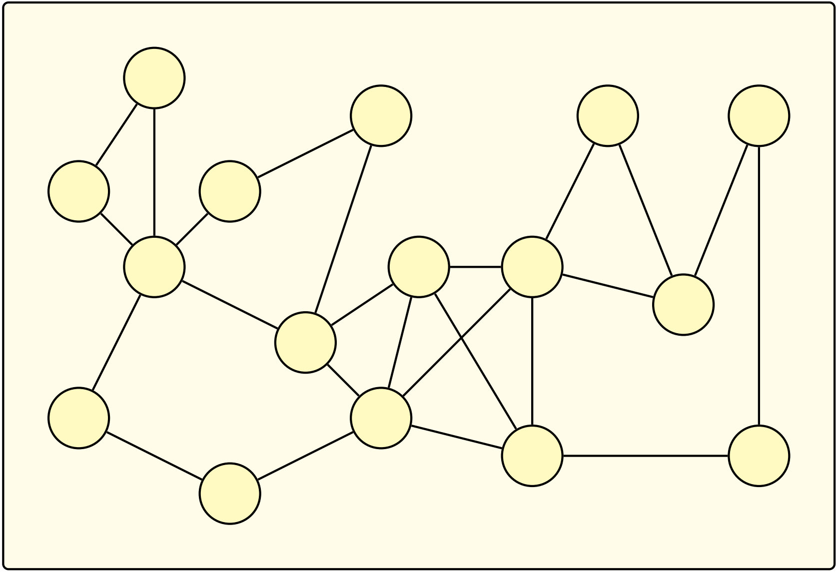


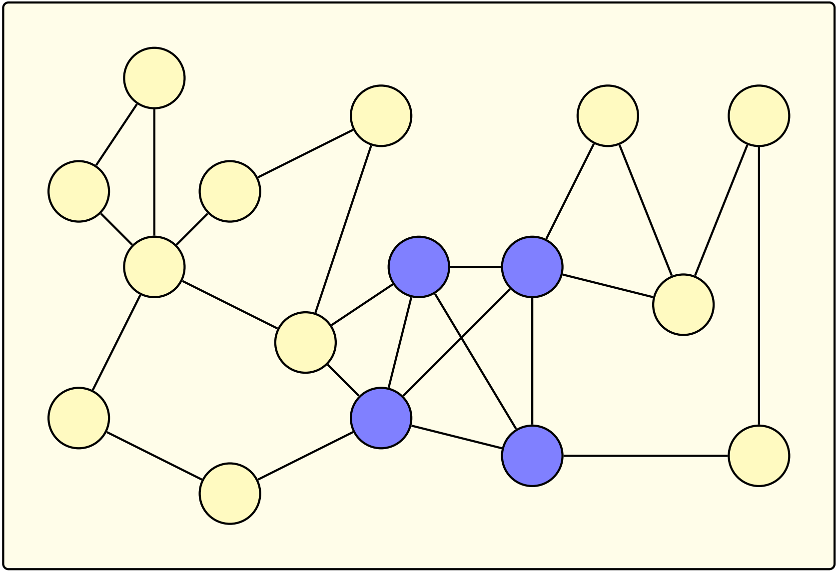
Vertex Cover: NP-complete

Vertex Cover is known to be *NP-complete*.

It is one of Karp's original 21 problems. Reduction from Chromatic Number (graph coloring).

It is easy to bring back to Satisfiability however, or at least 3-Satisfiability.





Practical solvability

CLIQUE is *also* NP-complete (reduction from SATISFIABILITY by Karp).

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LTL MODEL CHECKING is *PSPACE*-complete, but used in practice to verify hardware design

1. *Approximation*
2. *Randomization*
3. *“Islands of tractability”*
4. *Parallelization*
5. *Parameterization*

Approximation means accepting less than “perfect” solutions to reach a tractable algorithm.

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A algorithm is a c-approximation for a problem P if it computes an answer that is at most c times worse than the optimal.

VERTEX COVER has a 2-approximation in polynomial time.

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PTAS: Polynomial time approximation schemes are the centerpiece.

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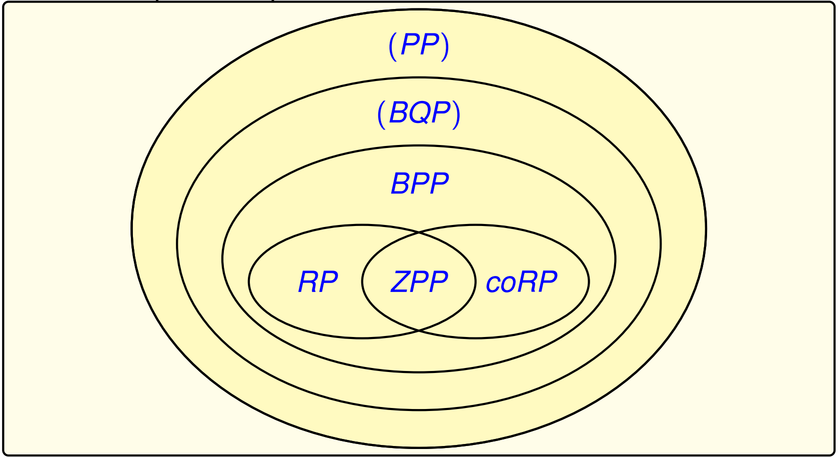
The zoo is quite complex:



Randomized is faster for many problems in P, but $P \stackrel{?}{=} BPP$

list, zpp, bqp, pp silly

3. Islands of Tractability

Not strictly defined, just a subset $P' \subset P$ of an NP-complete problem which is in P and has some argument why it is what is of *actual* interest.

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Fixing constants: SATISFIABILITY is in P if no clause has more than 2 literals.

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Easy but clumsy: parameterized complexity picks up from here

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The complexity theory is quite interesting, the **NC** (Nick's Class after Nick Pippenger) hierarchy defines how problems may be split

(log to c) n, n to k

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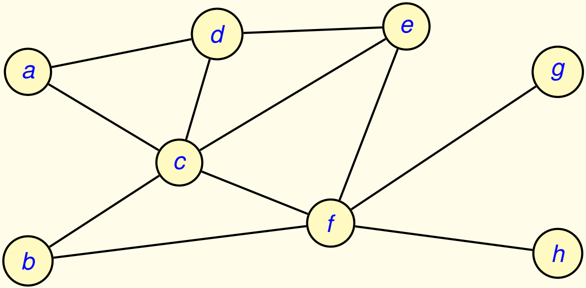
in general parallelization operates within P to an even greater extent than randomization:

- Most complex problems actually resist parallelization
- Even if an NP-complete problem can be parallelized this entails *increasing the amount of hardware exponentially* (unless P=NP)

(log to c) n, n to k

Part II: Fixed Parameter Tractability



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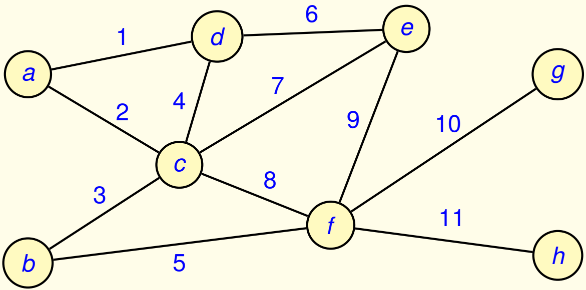
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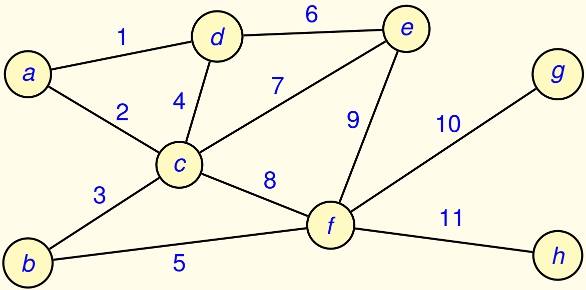
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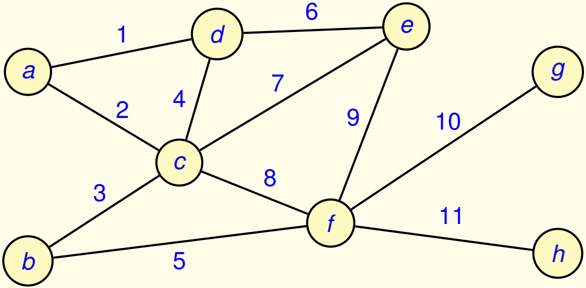
Solving 3-Vertex Cover



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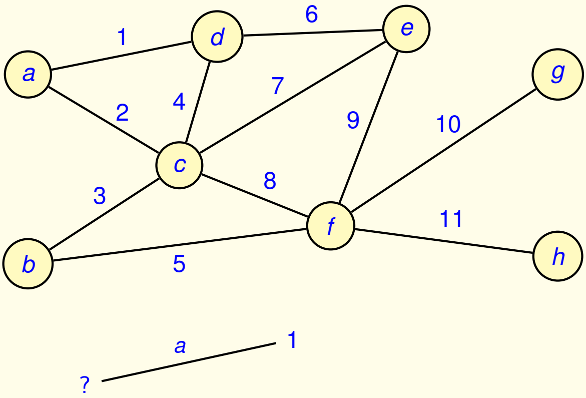

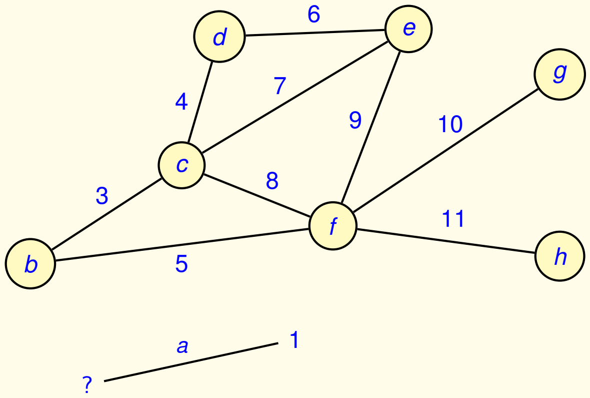
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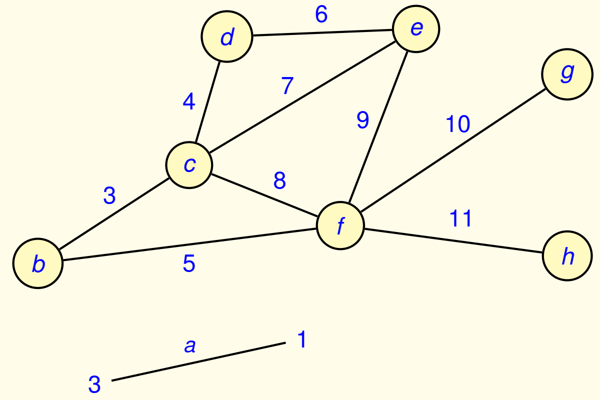

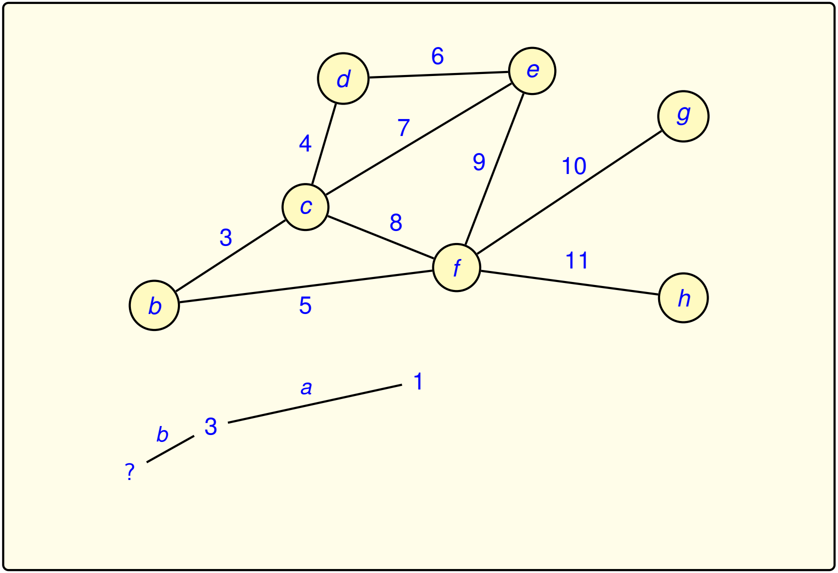

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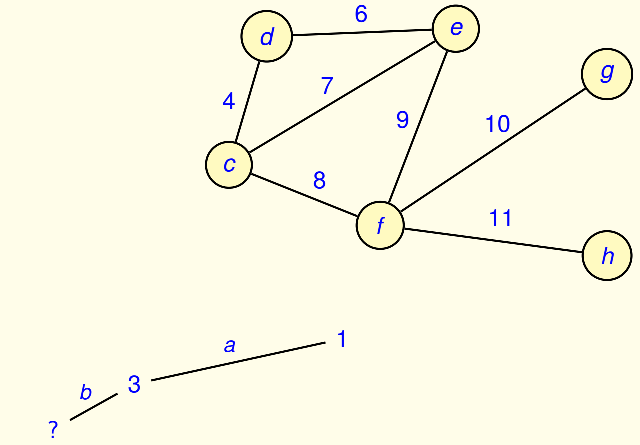

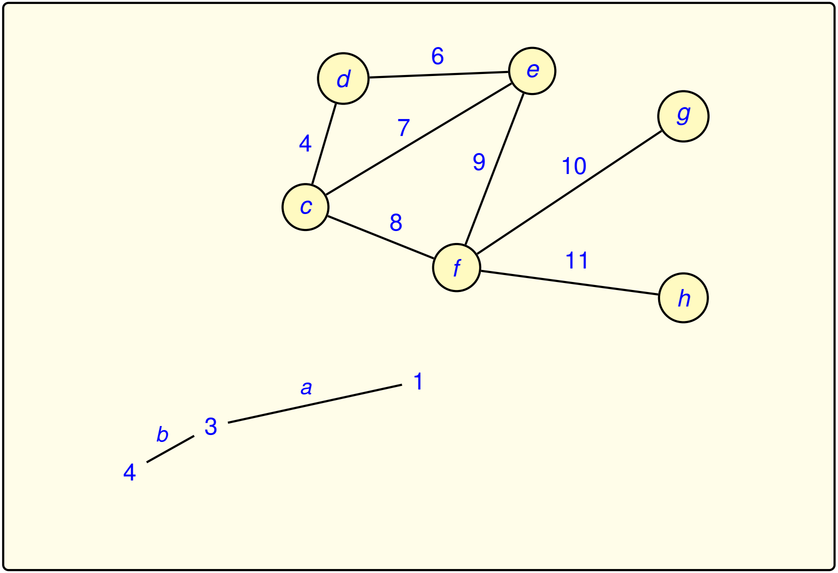

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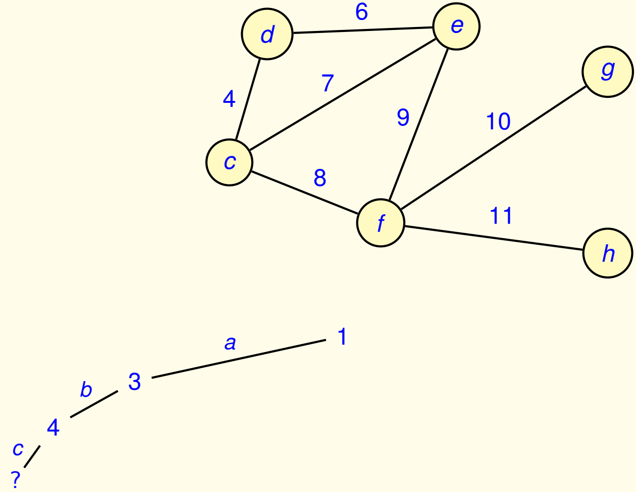

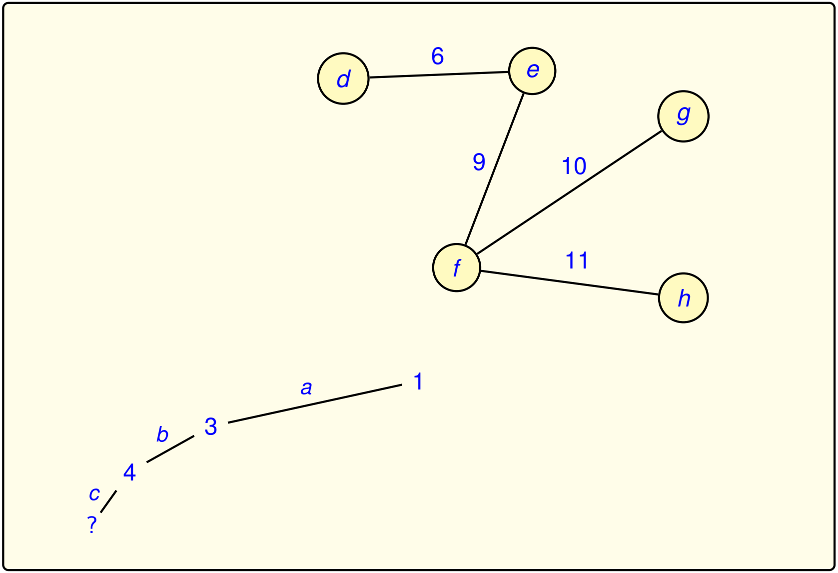
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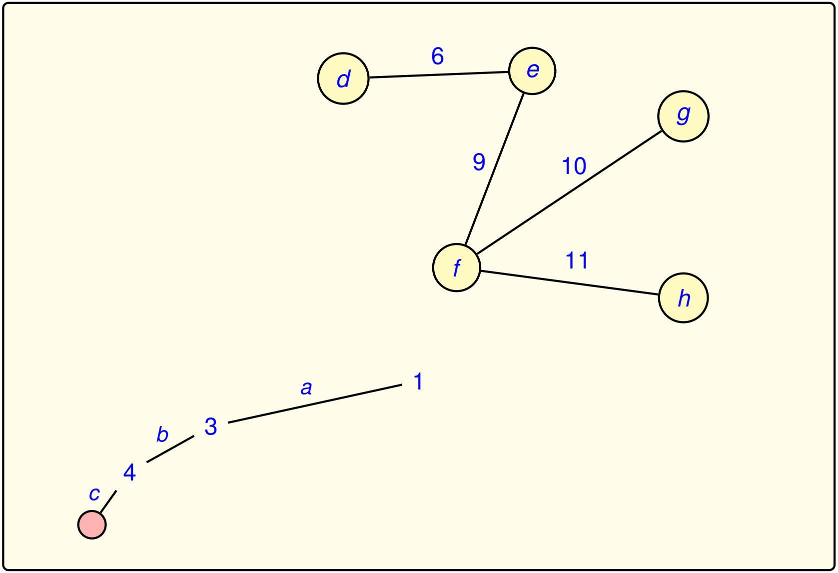
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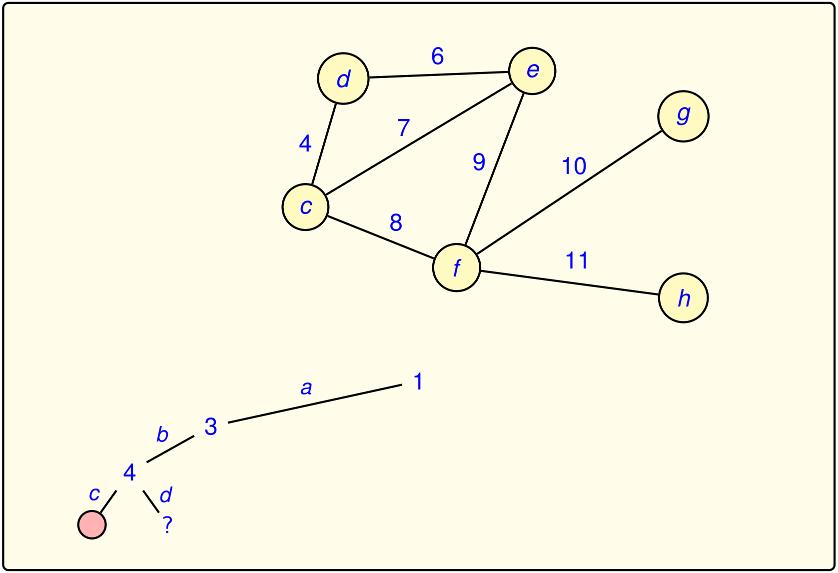

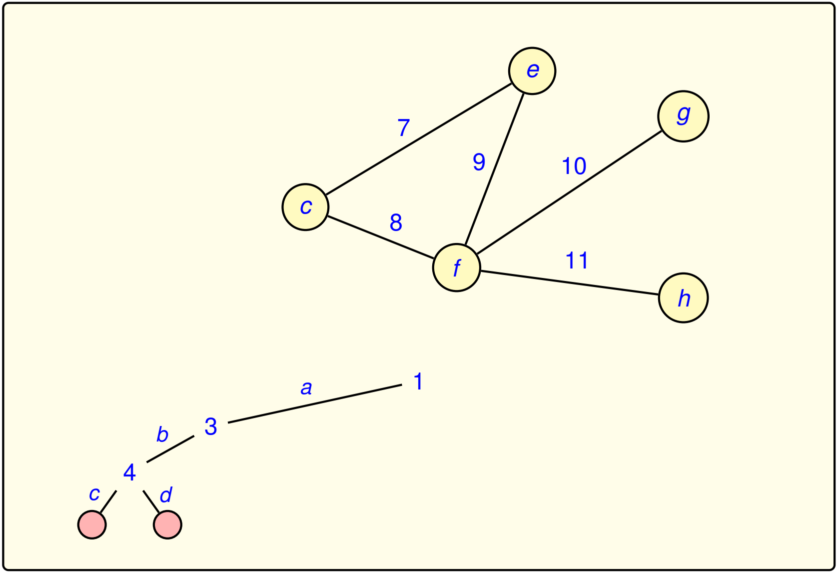
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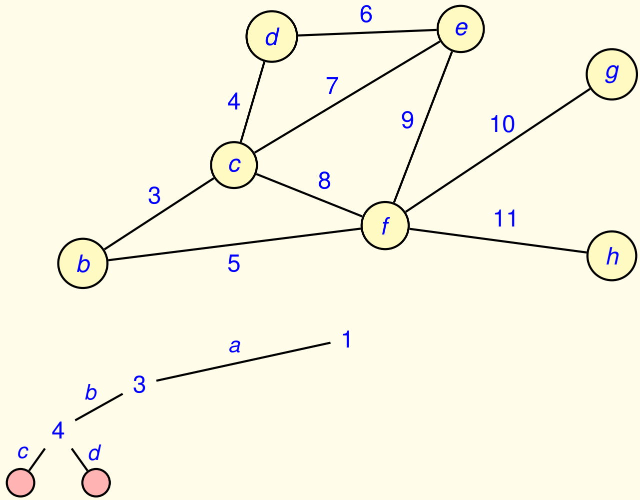
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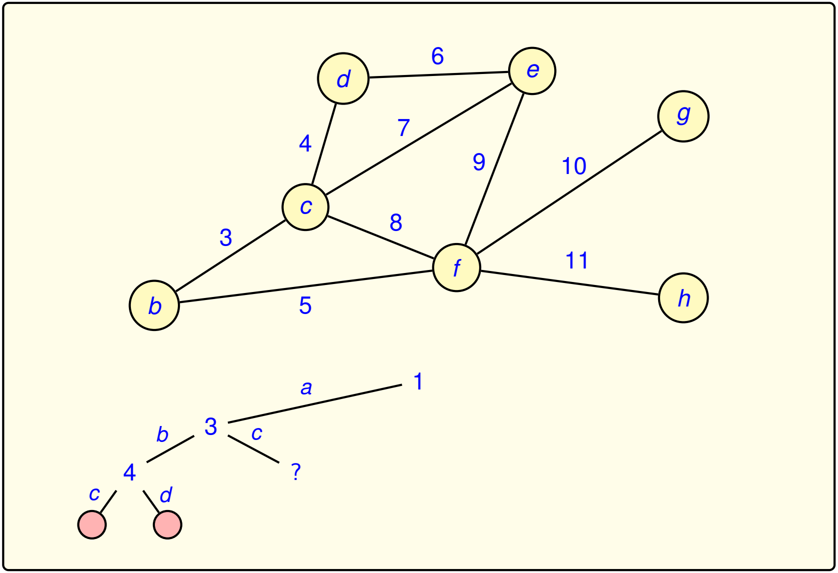
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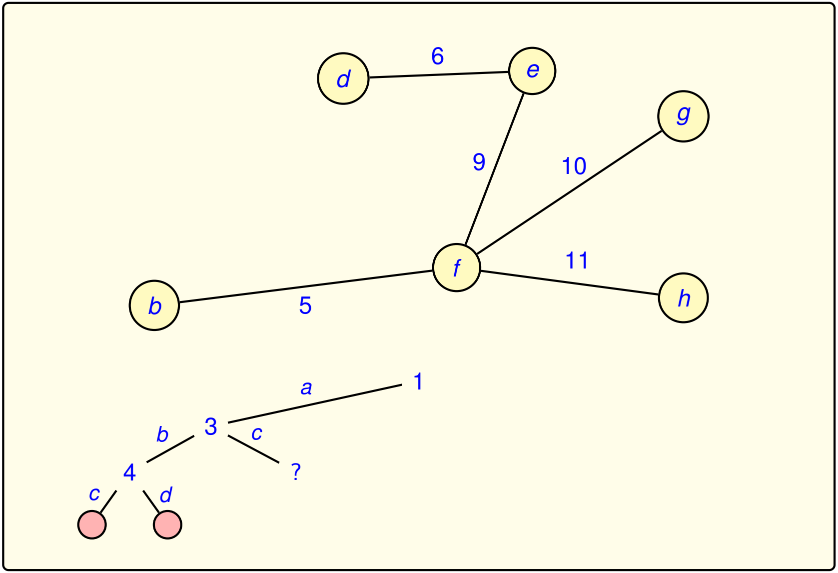

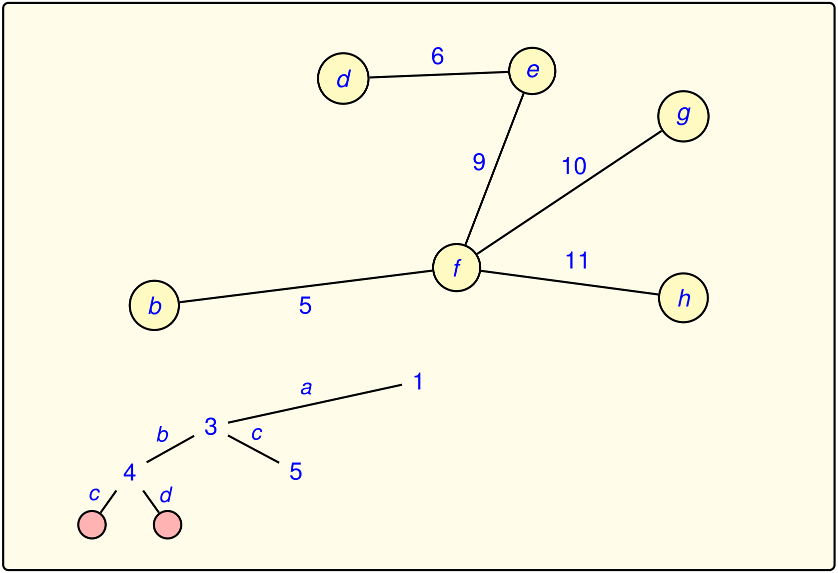

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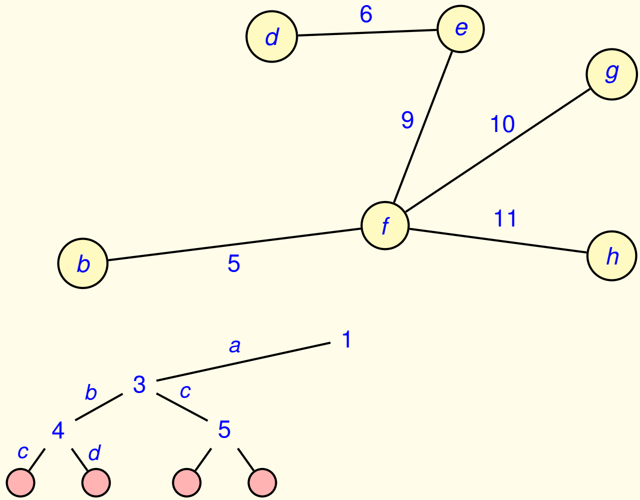
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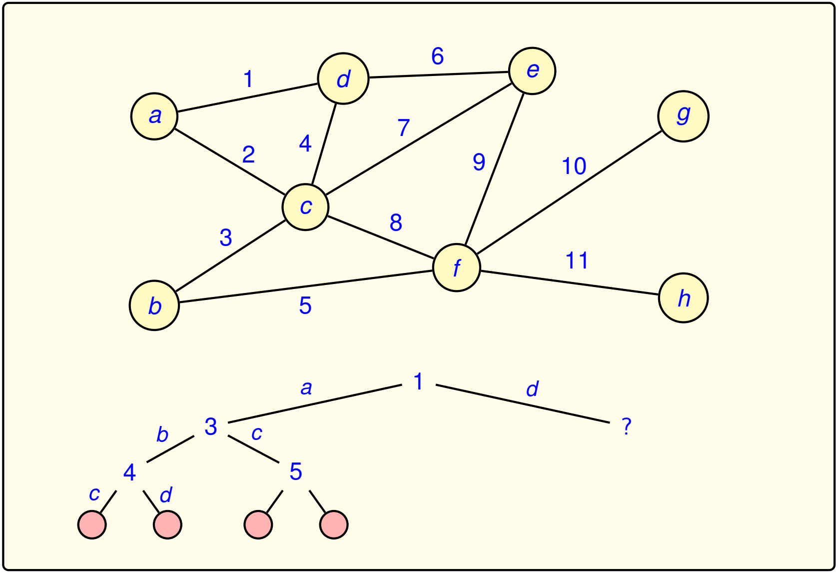

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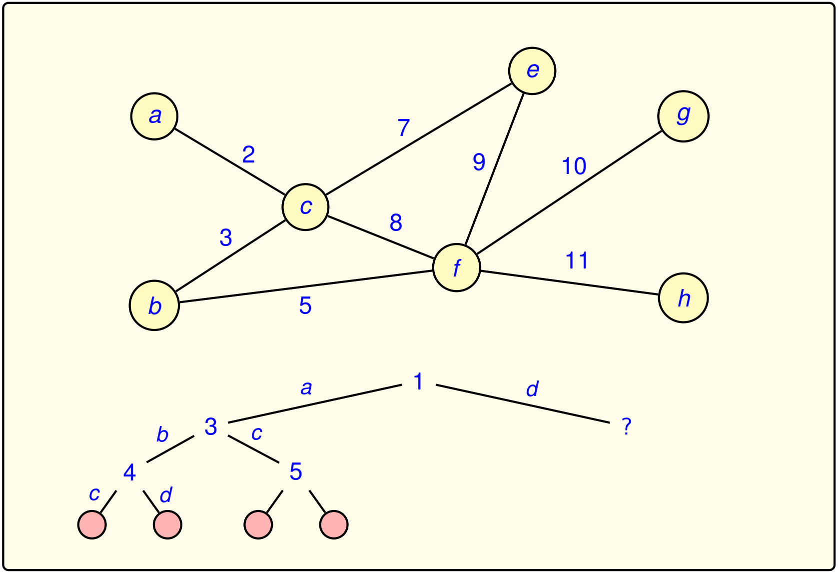
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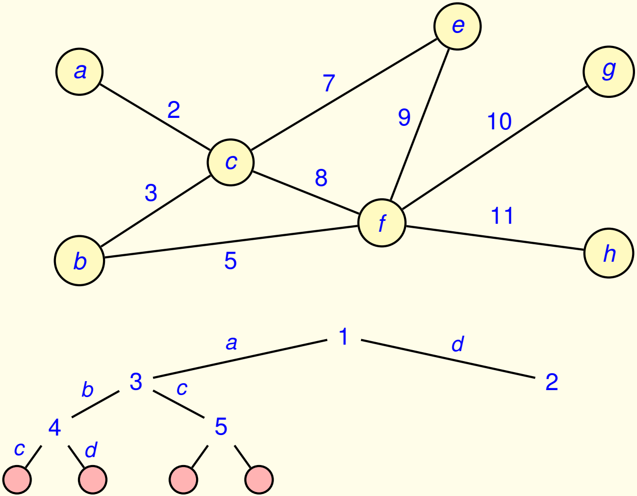

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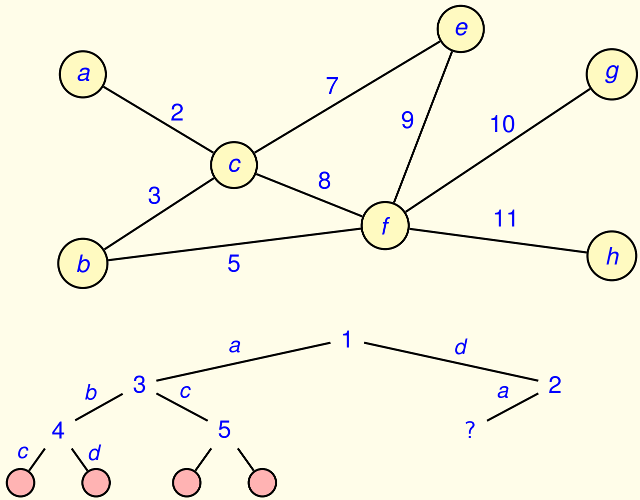

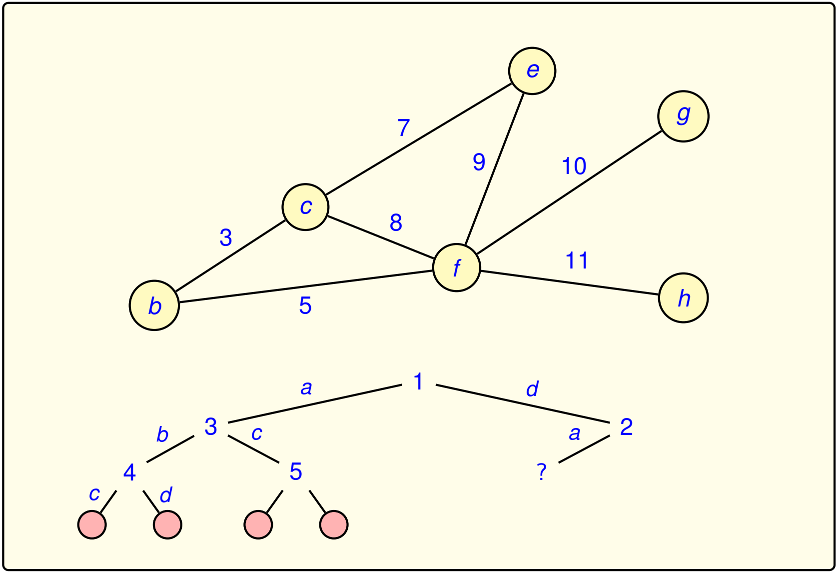
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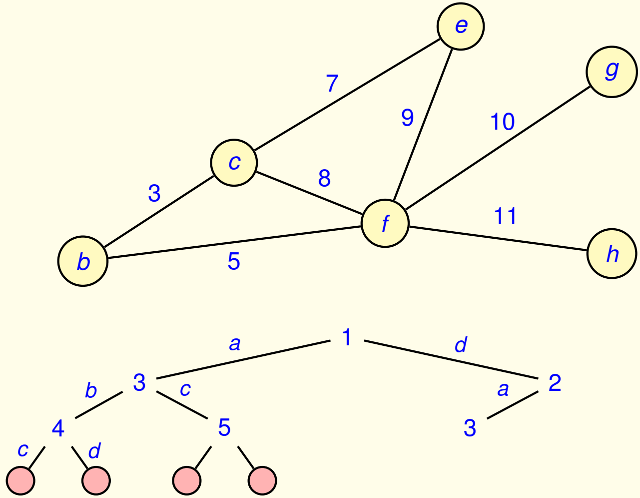
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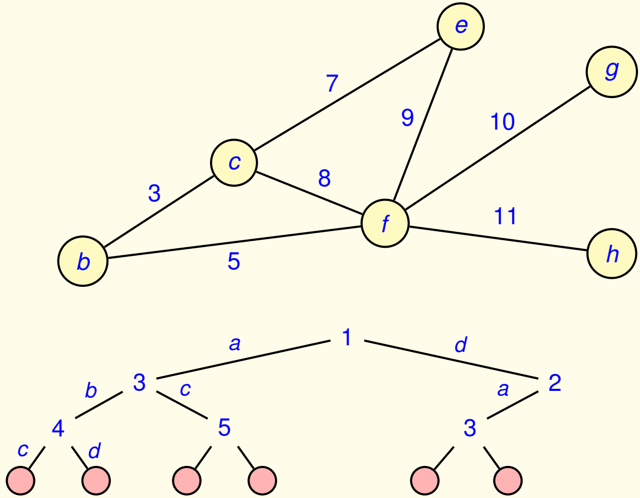
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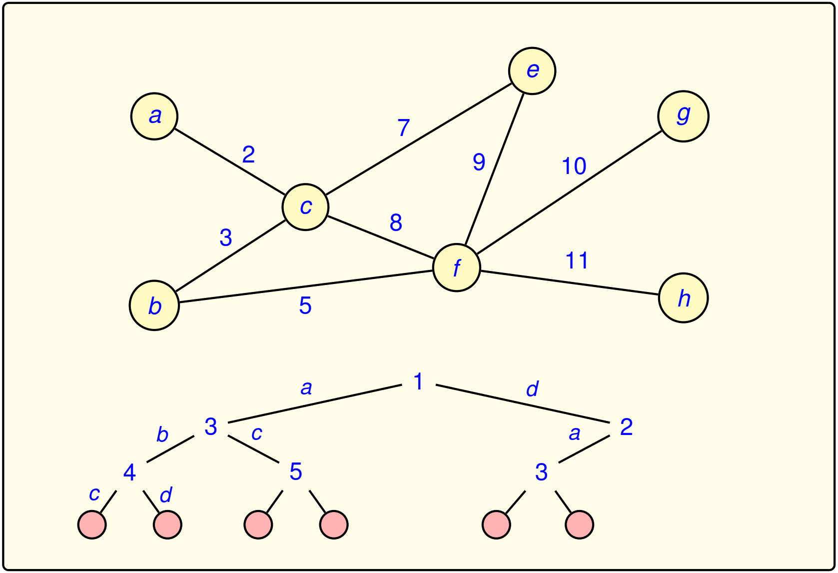

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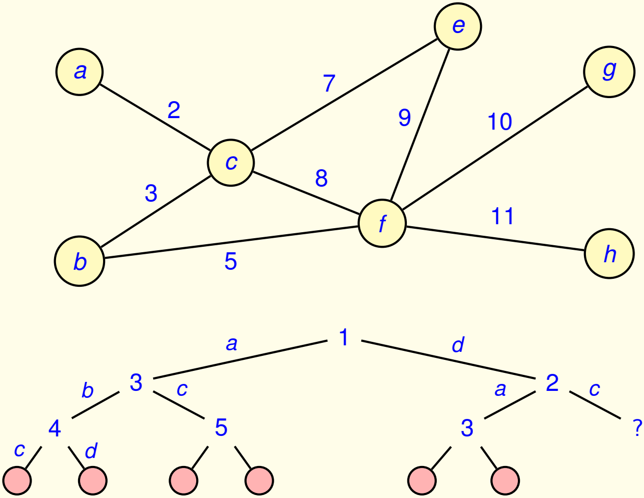

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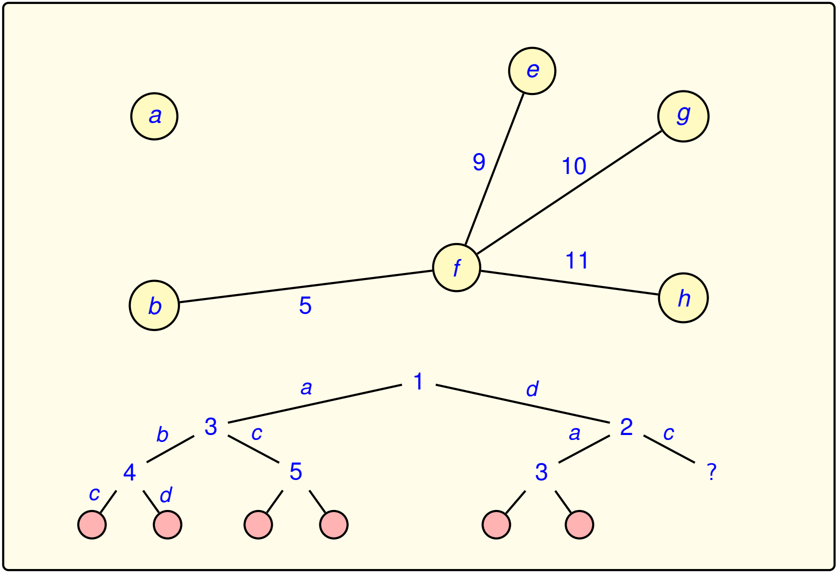

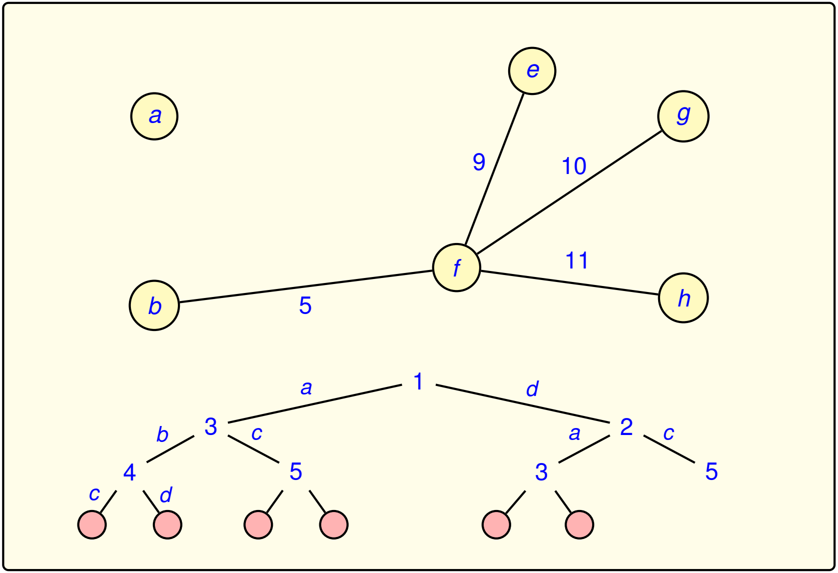
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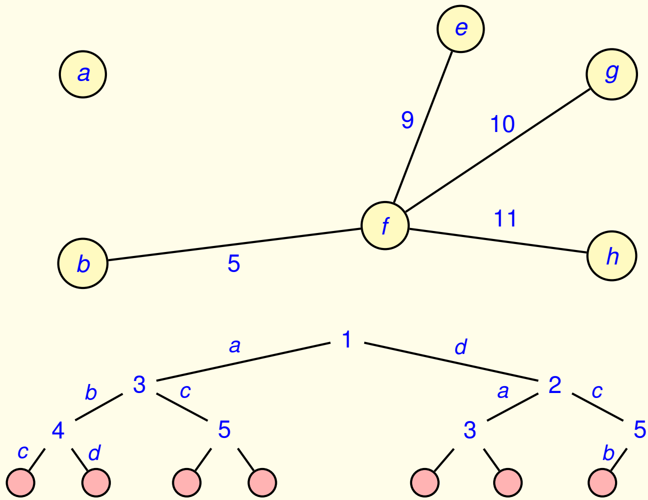
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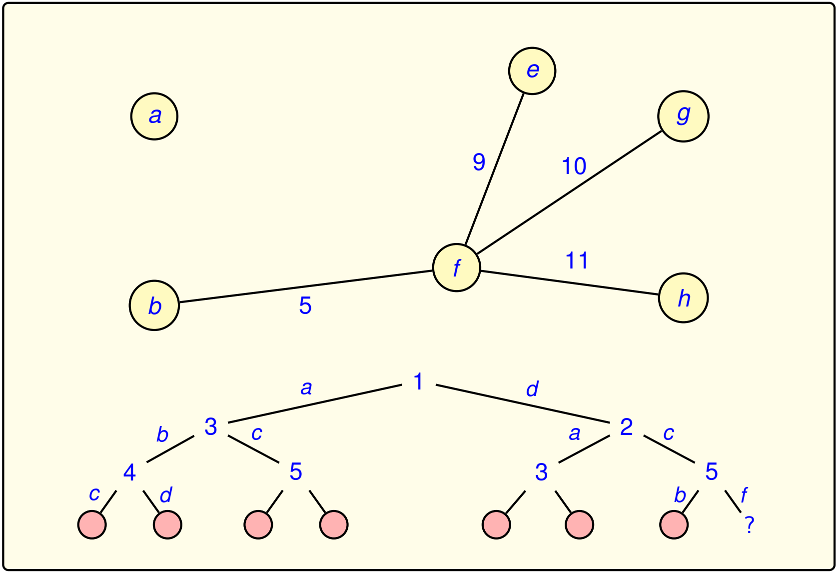

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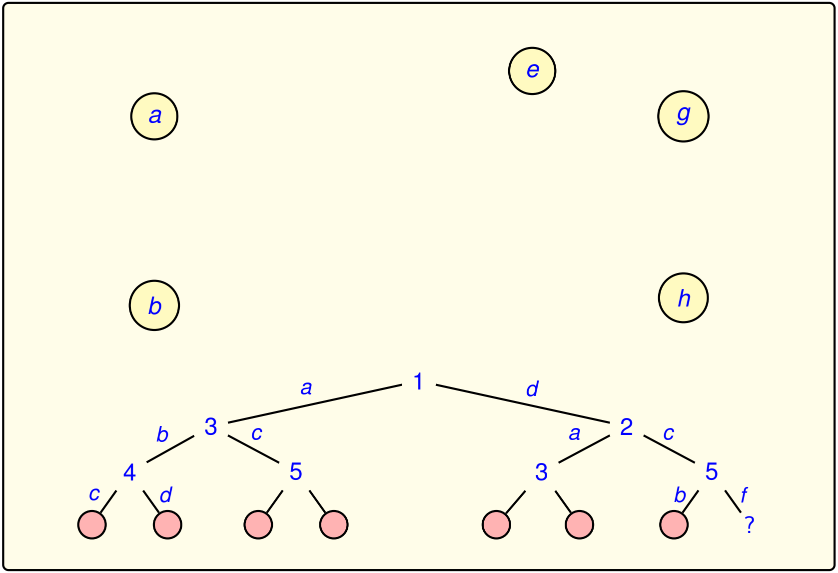

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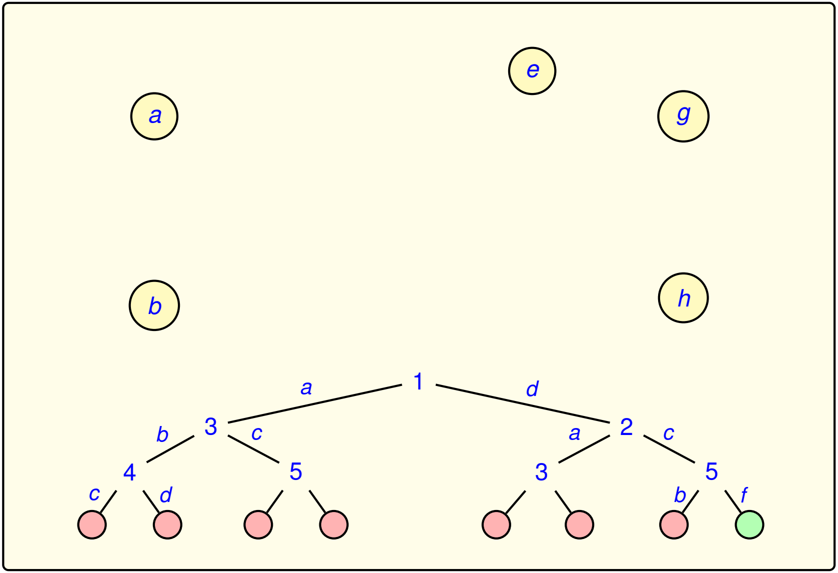

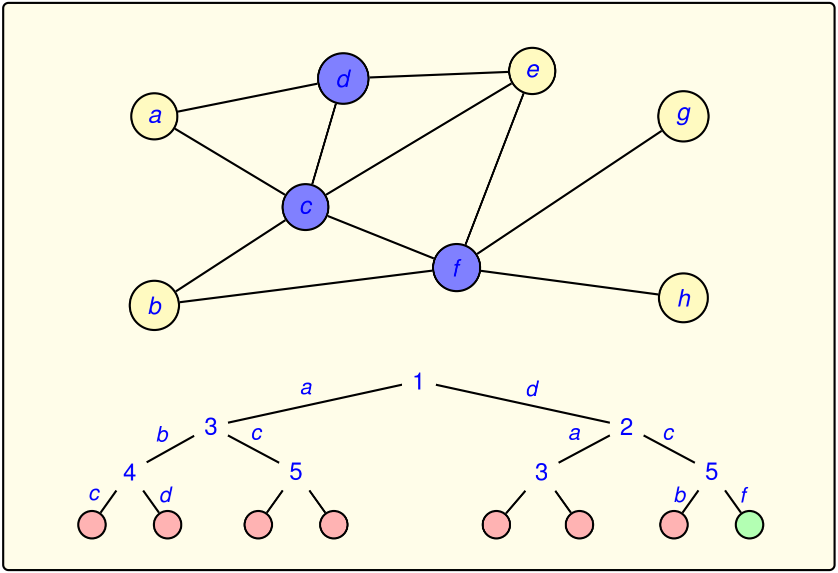
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We say that VERTEX COVER *parameterized by k* is *fixed-parameter tractable*.

Definition. Let Σ be a finite alphabet. A *parameterization* of Σ^* is a mapping $\kappa : \Sigma^* \to \mathbb{N}$ that is computable in polynomial time.

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Example. We can parameterize VERTEX COVER by setting

$$\kappa(G, k) = k.$$

Definition. Let Σ be a finite alphabet. A *parameterized problem* over Σ is a pair (Q, κ) consisting of

- a set $Q \subseteq \Sigma^*$ of strings over Σ, and
- a parameterization κ of Σ^*.

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- An algorithm is *FPT* w.r.t. κ if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ and a polynomial p such that for every $x \in \Sigma^*$, the algorithm, when given x, has running time at most
$$f(\kappa(x)) \cdot p(|x|).$$

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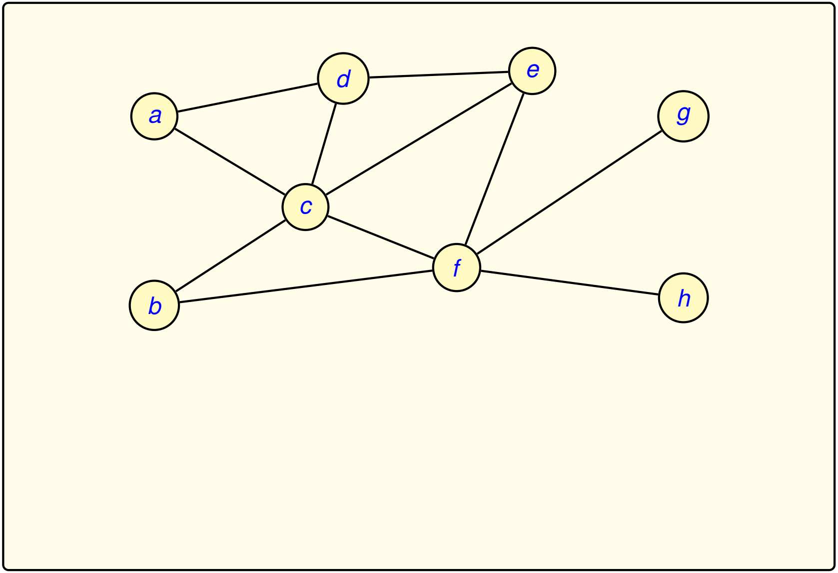
FPT is the class of all fixed-parameter tractable problems.

$\Rightarrow$

$k = 4$



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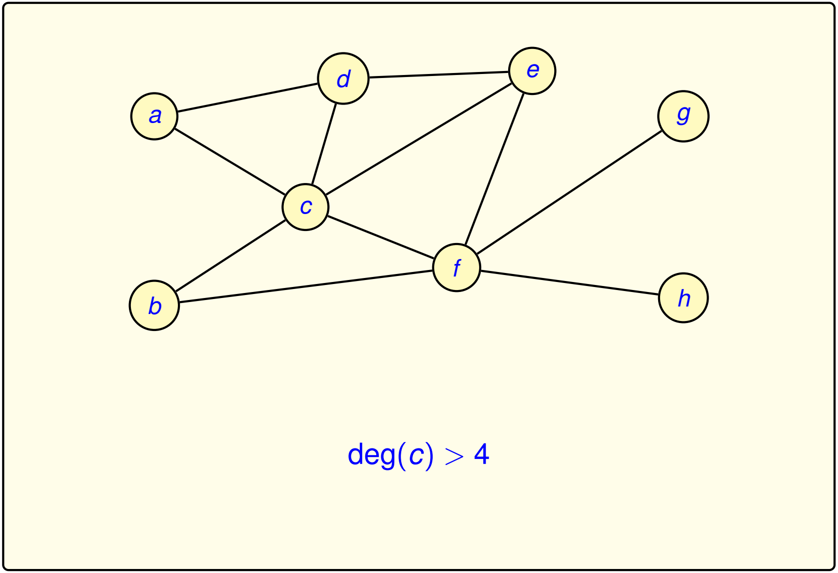
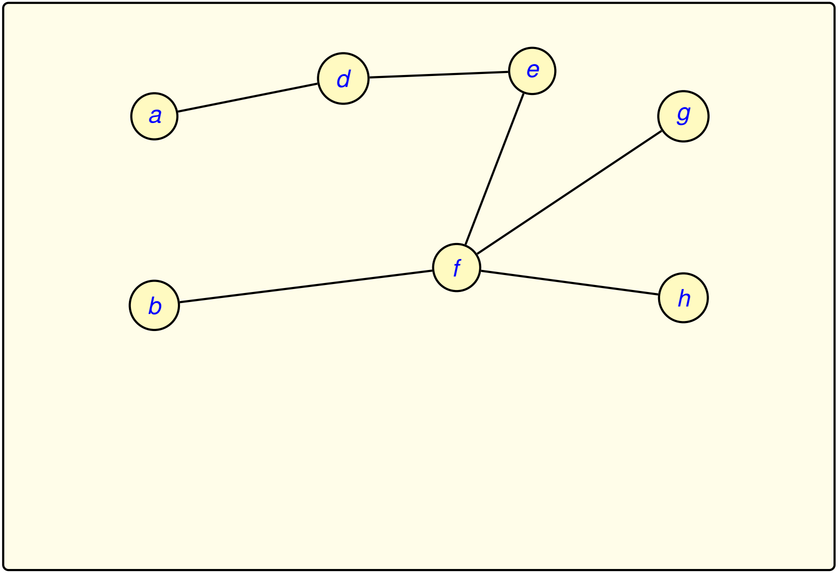


$k = 3$



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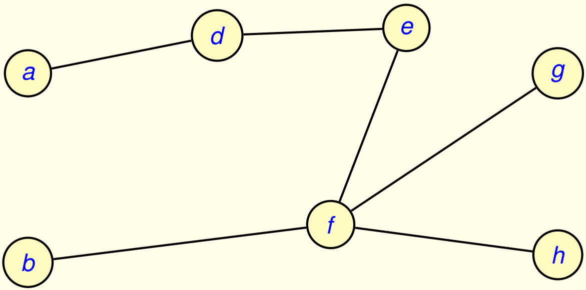
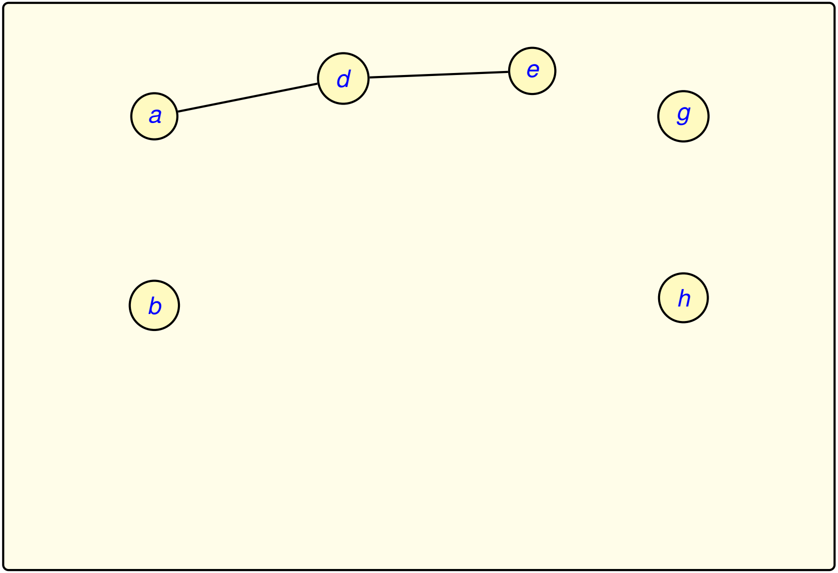


$\deg(f) > 3$

$k = 2$



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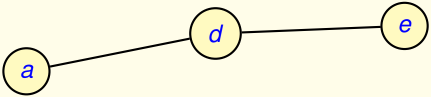

$|E| < 2^2$

More generally: *exhaustively applying the 3 reduction rules* transforms any VERTEX COVER instance into one where every vertex has a degree $2 \leq d \leq k$

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Combinatorics show that a k-coverable graph where all vertices have degree $2 \leq d \leq k$ cannot have more than k^2 vertices

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Combinatorics show that a k-coverable graph where all vertices have degree $2 \leq d \leq k$ cannot have more than k^2 vertices

A *polynomial procedure* making *the whole problem* work in terms of k!

Applying the bounded branching algorithm to a reduced graph gives us $\mathcal{O}(1.2738^k)k^2$

Let (Q, κ) be a parameterized problem over Σ.
A *kernelization* of (Q, κ) is a mapping

$$K : \Sigma^* \rightarrow \Sigma^*$$

such that

- $x \in Q \Leftrightarrow K(x) \in Q$,
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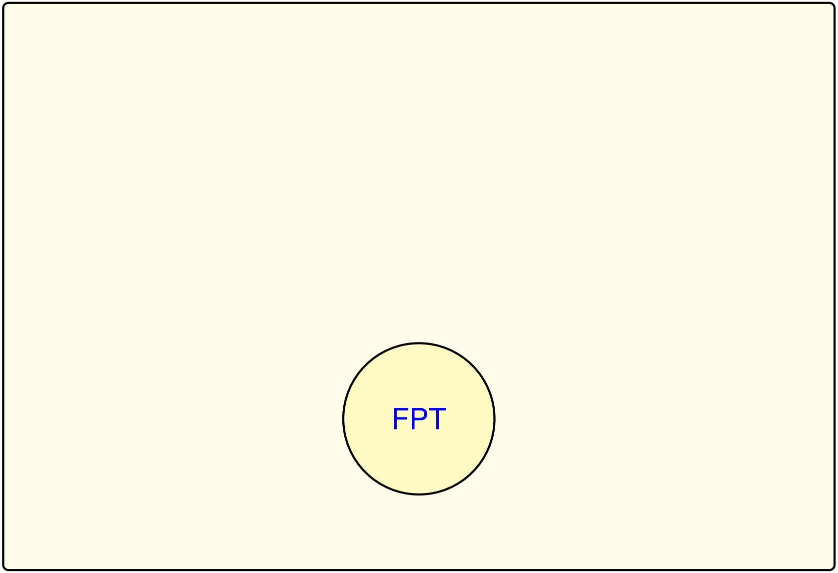
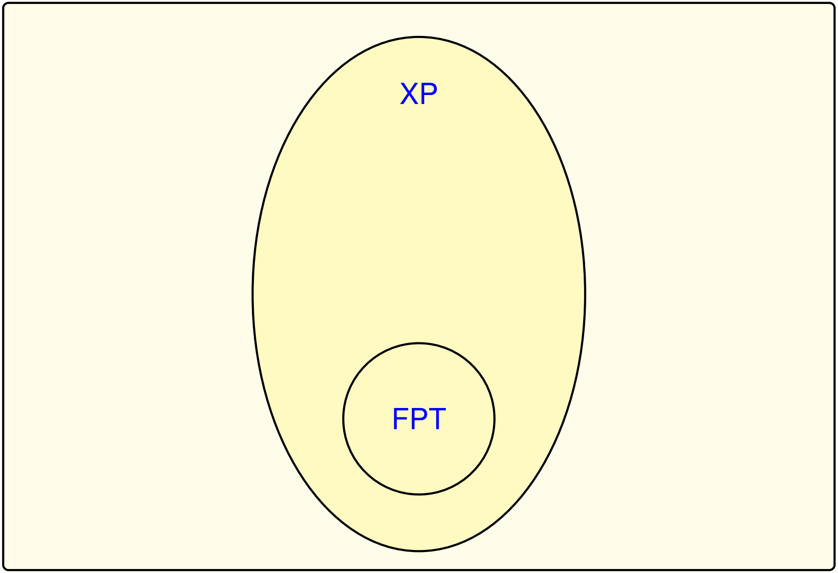
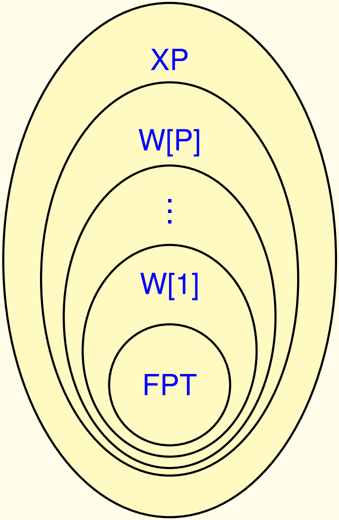

FPT


XP
FPT

The parameterized hierarchy



Part III: Proving Limited Parameterizability

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- For P and NP: any reduction which runs in P solves both
- For parameterized complexity more care is needed

Definition. Let (Q, κ) and (Q', κ') be two parameterized problems over Σ and Γ.

An *FPT-reduction* from (Q, κ) to (Q', κ') is a mapping $R : \Sigma^* \to \Gamma^*$ such that

1. $x \in Q \Leftrightarrow R(x) \in Q'$, for all $x \in \Sigma^*$,
2. R is *FPT-computable* w.r.t. κ, and
3. there is a computable function g such that $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$.

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The top of the hierarchy, XP, is also interesting

The class XP

Definition. Let (Q, κ) be a parameterized problem over Σ. Then (Q, κ) belongs to *XP* if there is a function $f : \mathbb{N} \to \mathbb{N}$ and an algorithm that decides Q and runs on input $x \in \Sigma^*$ in time

$$|x|^{f(\kappa(x))} + f(\kappa(x)).$$

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Recall *slicing* as an island of tractability technique

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Recall *slicing* as an island of tractability technique

This should illustrate why it can be viewed as crude compared to parameterized complexity: *it is the worst case of the hierarchy*

Conclusion

This completes the small tour of tractability hunting

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Good books to read:

Computational Complexity by *Christos H. Paradimitriou*

An excellent treatment of the central concepts in computational complexity.

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For now: *Thanks for listening and enjoy the rest of the course!*