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Teknisk beräkningsvetenskap II

Case Study II: Models of transport, waves, and shallow waters
Part 2: A sluice gate opens

In this lab, we will simulate the hydrodynamic effects (nonlinear wave motion) of opening a gate in a simplified model of a sluice. Section 1 derives the governing equation, the shallow-water equations in one space dimension, and the particular case we will study is described in section 2. Section 3 contains the studies you should perform, and section 4 a few implementation hints.

1 Modeling shallow channel flow

Shallow-water flow is a fluid-flow case in which the horisontal length scales are much larger than the vertical length scale. When using the appropriate approximations for this case, the water surface level h enters as an unknown function that is solved for together with the horisontal velocity. The shallow-water equations can be derived by depth-averaging a more general set of governing equations. However, we will derive them from scratch in under the following assumptions.

Assumptions.
1. The fluid has the constant density ρ (in kg/m^3).
2. The channel is narrow and shallow with a planar bottom.
3. Inertial forces dominate over viscous forces. In particular, the fluid's friction forces against the bottom and sides of the channel are ignored.

We assume that the bottom of the channel is located at $z = 0$ and that its width is d (Figure 1). The narrowness of d (Assumption 2) means that the x-direction dynamics dominates in the horisontal direction, and we therefore assume that the water level is constant in the y direction. (That is, we assume static conditions in the y direction). Thus, the area of each vertical cross section is given by

$$A(x,t) = h(x,t)d, \tag{1}$$

where $h(x,t)$ is the surface level. The shallow-water assumption means that we ignore the vertical dynamics, which means that we may assume that the velocity field is

$$\boldsymbol{u} = (u(x,t), 0, 0), \tag{2}$$

that is, we ignore the vertical velocity component and assume that the horisontal velocity component u is constant in the z- and y-direction. Alternatively, we can say that u represents the actual horisontal velocity averaged over $A(x,t)$. A second consequence of ignoring the vertical dynamics is that we assume a static pressure distribution in the vertical direction:

$$p(x,z,t) = \rho g(h(x,t) - z) \qquad \text{for } z \in [0, h(x,t)], \tag{3}$$

that is, the pressure at vertical level z equals the pressure given by the column of water above z (note that p is the atmospheric overpressure; that is, $p = 0$ at the water surface).

We will derive equations for the water surface level h and the horisontal velocity u as a function of x and t. The equations will follow from the laws of mass conservation and momentum balance together with assumption (3), which here acts somewhat like a constitutive relation that follows from the shallow-water assumption.

Now consider an arbitrary interval (a,b) on the x axis (Figure 1). The mass of fluid between vertical planes at the end points of the interval is

$$m(t) = \int_a^b \rho A\,dx. \tag{4}$$

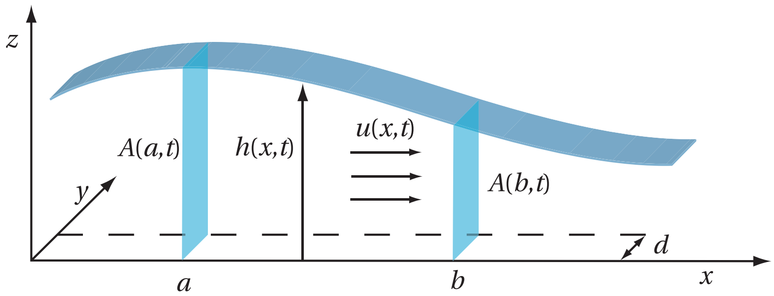

FIGURE 1. Shallow-water flow in a narrow channel of width d. The water surface level $h(x,t)$ and the horisontal velocity $u(x,t)$ are the unknown functions.

The mass flux (in kg/s) in the positive x direction through a vertical cross section at point x with area $A(x)$ is

$$f^{(m)}(x,t) = \rho u(x,t) A(x,t). \tag{5}$$

The law of mass conservation implies that the increase of the mass in the region is equal to the influx through the boundary at $x = a$ minus the outflux through the boundary at $x = b$,

$$\frac{dm}{dt} = f^{(m)}(a,t) - f^{(m)}(b,t) = -\int_a^b (f^{(m)})_x\, dx, \tag{6}$$

which by expressions (4) and (5) yields

$$\int_a^b (\rho A)_t\, dx = -\int_a^b (\rho u A)_x\, dx. \tag{7}$$

Dividing expression (7) with $d\rho$, using expression (1), that ρ is constant, and that interval (a,b) is arbitrary, we obtain the equation of mass conservation under the current assumptions,

$$h_t + (uh)_x = 0. \tag{8}$$

The momentum (sv. *rörelsemängd*) of a particle is its mass times its velocity. The momentum of a fluid per unit volume is in the general case $\rho \boldsymbol{u}$. Momentum, as opposed to mass, has a *direction*. In the general case, the momentum per unit volume is a vector field; here it is is a signed quantity, ρu, which is positive in the x-axis direction. Momentum is a conserved quantity, like mass, if there are no forces acting on the system. (This is Newton's first law). Otherwise, for a *fixed* amount of material, the change in momentum equals the applied forces. (For particles, this is Newton's second law). If the amount of material is not fixed, as in our case for the material in the region in (a,b), the *momentum balance law*, a generalization of Newton's second law, says that the change of momentum equals the net influx of momentum plus the applied forces,

$$\frac{dr}{dt} = f^{(r)}(a,t) - f^{(r)}(b,t) + f^{(p)}(a,t) - f^{(p)}(b,t) = -\int_a^b (f^{(r)} + f^{(p)})_x\, dx, \tag{9}$$

where r is the momentum, $f^{(r)}(x)$ the flux of momentum in the positive x-direction through a vertical plane at x, and $f^{(p)}(x)$ the force applied in the positive x-direction on a vertical plane at x.

In our case, the momentum of the region between vertical cross sections at $x = a$ and $x = b$ is

$$r(t) = \int_a^b \rho u A\, dx. \tag{10}$$

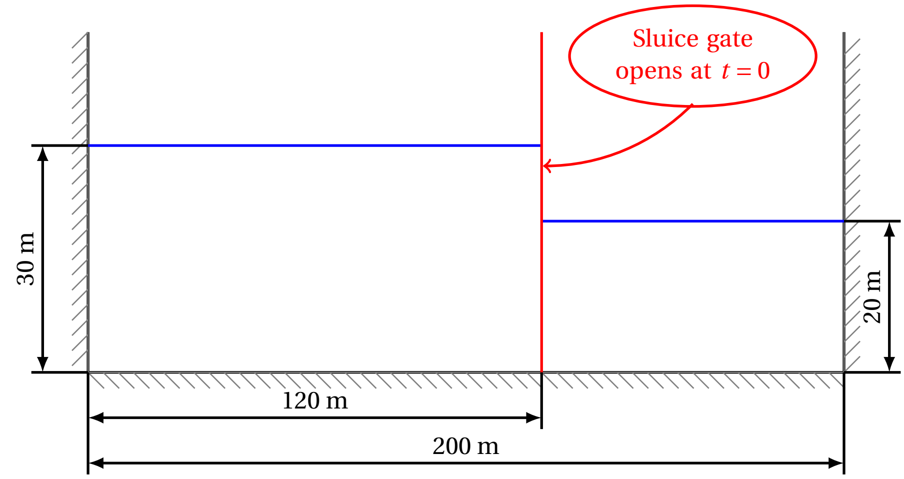

FIGURE 2. Sketch of the sluice setup before the gate opens.

The flux of momentum in the positive x-direction through a vertical cross section at point x with area $A(x,t)$ is

$$f^{(r)}(x,t) = \rho u(x,t)^2 A(x,t), \tag{11}$$

that is, the product of ρu, the momentum/m^3, and uA, the volume flux in m^3/s. Integrating the pressure (3) over $A(x,t)$ yields the force in the positive x-direction on a vertical cross section at point x,

$$\begin{aligned} f^{(p)}(x,t) &= \int_0^h \int_0^d p(x,z,t)\, dy\, dz = \int_0^h \int_0^d \rho g\,(h-z)\, dy\, dz. \\ &= -d\rho g \frac{(h-z)^2}{2}\bigg|_{z=0}^{h} = d\rho g \frac{h^2}{2}. \end{aligned} \tag{12}$$

Substituting the momentum expression (10), and the flux expressions (11), (12) into the momentum balance law (9) yields

$$\int_a^b (\rho u A)_t\, dx = -\int_a^b \left(\rho u^2 A + d\rho g h^2/2\right)_x dx. \tag{13}$$

Again dividing by $d\rho$ and using that interval (a,b) is arbitrary, we obtain the equation of momentum balance under the current assumptions,

$$(uh)_t + (u^2 h)_x + g(h^2/2)_x = 0. \tag{14}$$

Defining $q = uh$ (qd is the volume flux in m^3/s), we may write equations (8) and (14) as the following system of conservation laws,

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ q \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q \\ q^2/h + g h^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{15}$$

which are the shallow-water equations for a narrow channel.

2 Opening of a sluice gate

Now we are ready to look at the particular features of the problem we want to study. Consider the sluice portion (the lock) depicted in Figure 2. Here we are considering a lock consisting of two chambers of length 120 and 80 meters, respectively. Initially, the water height is 30 m in the large lock and 20 m in the smaller lock, at time $t = 0$, the gate between the two lock opens

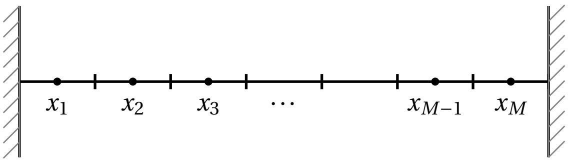

FIGURE 3. The positioning of the cell centers $x_i = i\Delta x/2$, $i = 1, \ldots, M$, and the cell interfaces $x_{i-1/2} = i\Delta x$, $i = 0, \ldots, M$.

and water is free to flow between the two chamber. In this, study we will simply model the opening of the lock by removing the complete gate. In reality, the gate opening is typically more controlled in that the gate is only partially opened at the bottom, but a simulation of this case would require a more complete model. In addition, we assume that the water is completely still at the instant the gate is removed, that is $u \equiv 0$ at $t = 0$, hence the initial condition for system (15) are

$$h(0,x) = \begin{cases} 30 & \text{if} \quad 0 \le x \le 120, \\ 20 & \text{if } 120 \le x \le 200, \end{cases} \qquad \text{and} \qquad q(0,x) = 0 \text{ for } x \in [0,200]. \tag{16}$$

To close the system, we need boundary conditions at $x = 0$ and $x = 200$. There is no flow of water across the chamber walls, so $u = 0$ on these boundaries, and hence

$$q(t,0) = q(t,200) = 0 \quad \forall t \ge 0. \tag{17}$$

3 Computer Exercises

1. Numerically solve system (15) with initial condition (16) and boundary condition (17) on a uniform grid using Lax–Friedrichs method. Solve the problem for a sufficiently large time so that the effect of the walls at the sides of the sluice is clear.
2. Estimate the shock speed of the wave travelling from the gate ($x = 120$) towards the right wall ($x = 200$). Here, we are only interested in the speed of the initial shock. It is thus sufficient to solve the problem until the shock wave has moved a certain distance, but not yet reached the right wall.
3. Vary the time and space discretizations. Does this affect the estimated shock speed?
4. Vary the inital condition, that is, change the water height in the two basins. Try for example, 20 m (left basin) and 10 m (right basin) as well as 30 m (left basin) and 15 m (right basin). Does this affect the estimated shock speed?

4 Implementation instructions and hints

4.1 General instructions and hints for all exercises

- Let the number of cells M and the time–space step ratio $\lambda = \Delta t/\Delta x$ be input parameters.
- For this problem, we need to set $\lambda \lesssim 0.05$ to avoid stability problems.
- Remember that, as illustrated in Figure 3, x_i, $i = 1,2,\ldots,M$ are the cell *centers* located at $\Delta x/2$, $3\Delta x/2, \ldots, (2M-1)\Delta x/2$ and that u_i^n, $i = 1,2,\ldots,M$ are corresponding cell averages. The cell *interfaces* are located at $x_{i+1/2} = i\,\Delta x$, $i = 0,1,\ldots,M$.
- The plotting routine `plotfvm`, available at the course homepage,[1] can be used to plot piecewise-constant functions. Routine `plotfvm` takes two input arguments, `x` and `u`, where `x` is an $M+1$ vector that contains the cell interface points $x_{1/2}, \ldots, x_{M+1/2}$, and `u` is an M vector containing cell averages $u_1, u_2, \ldots, u_M$. Note that the lengths of `x` and `u` are different!

[1] `http://www.cs.umu.se/kurser/5DV123/HT11/case_studies.php`

- The family of finite volume schemes that we consider in this course have the general form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\Big(F_{i+1/2}^n - F_{i-1/2}^n\Big), \tag{18}$$

  where u_i^n denotes the cell average in cell i at time t_n.

- Recall that the numerical flux function for the Lax–Friedrichs method is:

$$F_{i+1/2}^n = \frac{1}{2}\left(f(u_{i+1}^n) + f(u_i^n)\right) - \frac{\Delta x}{2\Delta t}\left(u_{i+1}^n - u_i^n\right). \tag{19}$$

  By inserting flux (19) in the general scheme (18), we obtain

$$u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right). \tag{20}$$

- We need to impose boundary conditions at $x = 0$ and $x = 200$. We will achieve this by modifying the fluxes $F_{1/2}^n$ and $F_{M+1/2}^n$. Consider update equation (18) for $i = 1$,

$$u_1^{n+1} = u_1^n - \frac{\Delta t}{\Delta x}\Big(F_{3/2}^n - F_{1/2}^n\Big) \tag{21}$$

  Here, we use the normal Lax–Friedrichs flux (19) to compute the right flux $F_{3/2}^n$ but substitute the normal left flux $F_{1/2}^n$ by imposing $q = 0$ at $x = 0$ and by approximating h at $x = 0$ by the cell average in the first cell. Under these conditions, update (21) becomes

$$\begin{aligned}
u_1^{n+1} &= u_1^n - \frac{\Delta t}{\Delta x}\Big(F_{3/2}^n - F_{1/2}^n\Big) \\
&= u_1^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left(f(u_2^n) + f(u_1^n)\right) - \frac{\Delta x}{2\Delta t}\left(u_2^n - u_1^n\right) - \begin{bmatrix} 0 \\ 0 + g(h_1^n)^2/2 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(u_1^n + u_2^n\right) - \frac{\Delta t}{\Delta x}\left(\frac{1}{2} f(u_2^n) + \frac{1}{2}\begin{bmatrix} q_1^n \\ (q_1^n)^2/h_1^n + g(h_1^n)^2/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 + g(h_1^n)^2/2 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(u_1^n + u_2^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_2^n) - \begin{bmatrix} -q_1^n \\ -(q_1^n)^2/h_1^n + g(h_1^n)^2/2 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(u_1^n + u_2^n\right) - \frac{\Delta t}{2\Delta x}\Big(f(u_2^n) - f(-u_1^n)\Big)
\end{aligned} \tag{22}$$

  By using a similar argument at $x = 200$, we get the following update for the averages in the last cell

$$u_M^{n+1} = \frac{1}{2}\left(u_{M-1}^n + u_M^n\right) - \frac{\Delta t}{2\Delta x}\Big(f(-u_M^n) - f(u_{M-1}^n)\Big). \tag{23}$$

- Define the initial water height in the right basin as well as the difference in height between the two basins be variables in your code or input parameters to your function.

- Let `base` denote the initial water height in the right basin and `increase` denote the height difference. Given the vector `xm` of the cell center positions, the initial condition for the water height can in Matlab be written as

```
base+increase*(xm<=120);
```

- With `base` and `increase` as above, the shock position can be found by

```
delta_x*find(h>base+increase/4,1,'last');
```

  where `h` is the vector with the cell averaged water heights and `delta_x` is the space step. Read the documentation (`doc find`) for more information of the very useful command `find`!