Review: Porous Media Flow

Martin Berggren

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Porous media

Materials with **microstructure** consisting of a mixture of a solid **matrix** (sv. *fastsubstans*) and **void** or pores (sv. *hålrum*). We assume that

- the matrix is undeformable,
- the pores are interconnected,
- **adhesive** and **capillary** forces dominates over inertia when a fluid flows through the pores

To simulate large-scale effects of fluid flow through porous media, a **continuum approximation** of the microscopic properties is typically used.

Porous ceramic with interconnected pores that vary in size from 5–500 µm. (Source: Wikimedia commons).

Porosity and REV

- $B_d(\boldsymbol{x}_0)$: Ball of diameter d centered at $\boldsymbol{x}_0$ with volume $|B_d(\boldsymbol{x}_0)|$
- Void indicator:
$$\gamma(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \text{void}, \\ 0 & \boldsymbol{x} \in \text{matrix} \end{cases}$$
- Def: The **porosity** of the material at $\boldsymbol{x}_0$ is the relative volume of the void:
$$\phi(\boldsymbol{x}_0) = \frac{1}{|B_d(\boldsymbol{x}_0)|} \int_{B_d(\boldsymbol{x}_0)} \gamma(\boldsymbol{x})\, dV$$
- d^*: smallest value for which $\phi(\boldsymbol{x}_0)$ as a function of d stabilizes
- $B_{d^*}(\boldsymbol{x}_0)$: **Representative Elementary Volume** (REV)
- A REV corresponds to a **point** in the continuum model
- The porosity is **homogeneous** if $\phi(\boldsymbol{x}_0)$ is the same for each $\boldsymbol{x}_0$. Otherwise the porosity is **heterogeneous**.

Mass conservation

- **Control Volume**: V. We assume $|V| \gg |\text{REV}|$
- ρ: density of the fluid in the void (in kg/m^3)
- Mass of fluid in the pores of V: $m = \int_V \rho\phi\, dV$
- S: a **surface** with normal vector $\boldsymbol{n}$. (Dimensions larger than REV)
- Total mass flux through S: $Q_m = \int_S \rho\, \boldsymbol{n}\cdot\boldsymbol{u}\, dS$ (in kg/s), where

  $\boldsymbol{u}$: the **apparent velocity** or **Darcy velocity** (sv. *makroskopisk medelhastighet*)
- The apparent velocity is an averaged velocity that gives the same flux as the real velocity
- $Q_m > 0$, means net flux **out** of from a closed volume with boundary S

Mass conservation

- Assume that all pores are filed with one fluid phase
- Changes in the mass of the fluid in a control volume V can only occur due to net in/outflux through S:

$$\underbrace{\frac{d}{dt}\int_V \rho\phi\, dV}_{\text{mass increase}} = \underbrace{-\int_S \boldsymbol{n}\cdot(\rho\boldsymbol{u})\, dS}_{\text{influx of material}} = \text{[Divergence Theorem]}$$

$$= -\int_V \nabla\cdot(\rho\boldsymbol{u})\, dV$$

- V arbitrary implies that

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

- Special case: constant density. The fluid is then **incompressible** (e.g. water) and the apparent velocity field is **divergence free**:

$$\nabla\cdot\boldsymbol{u} = 0$$

Darcy's law

$$\boldsymbol{u} = -\frac{K}{\mu}(\nabla p - \rho g \boldsymbol{e}_g)$$

- K: permeability (in m^3). Measures the medium's ability for fluid flow.
- μ: dynamic viscosity (in Pa s). Measures the fluid's inner friction.
- $\boldsymbol{e}_g$: unit vector in direction of gravity

- A **constitutive relation** that holds for many materials
- First shown empirically (Henry Darcy 1856). Can be derived by so-called homogenization of the Navier–Stokes equations
- Above form assumes **isotropic** material: the same permeability properties in each direction
- For **anisotropic** media, the scalar K will be replaced by a 3-by-3 symmetric positive-definite matrix.

Darcy's law + mass conservation for constant ρ

Assume $-\boldsymbol{e}_g = \boldsymbol{e}_y = (0,1,0) = \nabla y$. Darcy's law:

$$\boldsymbol{u} = -\frac{K}{\mu}(\nabla p - \rho g \boldsymbol{e}_g) = -\frac{K}{\mu}(\nabla p + \rho g \nabla y)$$

$$= -\frac{K\rho g}{\mu}\nabla\left(\frac{p}{\rho g} + y\right) = -\kappa\nabla(\hat{p} + y) - \kappa\nabla h$$

where

- $\kappa = K\rho g/\mu$: **hydraulic conductivity** (in m/s)
- $\hat{p}$: **pore gauge pressure** (in m). (The pressure in m water column).
- $h = \hat{p} + y$: the **pressure head** (in m)

- Together with the incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$, we find

$$-\nabla\cdot(\kappa\nabla)h = 0$$