# Review: conservation laws and Finite Volume Methods

Martin Berggren

14 november 2011



## Conservation laws

*Def.:* A (scalar) **conservation law** is a partial differential equation of the type

$$\frac{\partial \lambda}{\partial t} + \nabla \cdot \boldsymbol{f}(\lambda) = 0$$

$\lambda$: conserved quantity. Examples: density of mass, momentum (sv. *rörelsemängd*), or energy, concentration of a chemical compound.

$\boldsymbol{f}$: flux function; simplest example: $\boldsymbol{f} = \boldsymbol{u}\lambda$. The flux function is a *nonlinear* function of the conserved quantity in many interesting cases.

In integral form: for any control volume $V$ holds

$$\frac{d}{dt}\int_V \lambda \, dV + \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{f}(\lambda) \, dS = 0$$



## Conservation laws in one space dimension

Differential form:

$$u_t + f(u)_x = 0; \tag{1}$$

Integral form:

$$\frac{d}{dt}\int_a^b u \, dx + f(u(b,t)) - f(u(a,t)) = 0, \tag{2}$$

for any interval $(a, b)$.

Equation (1) is written in **conservative form**.
If everything is smooth, the chain rule yields that equation (1) also can be written in the **primitive form**

$$u_t + f'(u)u_x = 0;$$



## The transport equation

Simplest example of 1D conservation law, the **transport equation**

$$u_t + cu_x = 0$$

Expresses **transport** of a function in the positive $x$-axis direction and has the general solution

$$u(x,t) = f(x - ct)$$

for a given (differentiable) function $f$.

## Characteristics for the transport equation

The **characteristics** of the transport equation $u_t + cu_x$ are the lines in the $(x, t)$ plane satisfying the equation $x - ct = x_0$,

The characteristic curves (lines) in parametric form: $(x_0 + ct, t) = (X(t), t)$, where $X(t)$ is the solution to

$$\dot{X}(t) = c \qquad t > 0,$$
$$X(0) = x_0.$$

*The solution to the transport equation is constant along the characteristics*

$$\begin{aligned}\frac{d}{dt}u(X(t), t) &= u_t(X(t), t) + u_x(X(t), t)\dot{X}(t)\\ &= u_t(X(t), t) + cu_x(X(t), t) = 0\end{aligned}$$



## Characteristics for nonlinear conservation laws

Smooth solutions to nonlinear conservation laws are also constant along each characteristic curve. To see this, assume that $u$ is a smooth solution to

$$u_t + f(u)_x = u_t + f'(u)u_x = 0.$$

Define the characteristic curves $(X(t), t)$ where $X(t)$ solves the nonlinear ODE

$$\begin{aligned}\dot{X}(t) &= f'(u(X(t), t)) \quad t > 0,\\ X(0) &= x_0.\end{aligned} \tag{3}$$

Then $u(X(t), t)$ is constant:

$$\begin{aligned}\frac{d}{dt}u(X(t), t) &= u_t(X(t), t) + u_x(X(t), t)\dot{X}(t)\\ &= u_t(X(t), t) + f'(u(X(t), t)u_x(X(t), t) = 0\end{aligned}$$

The right-hand side of (3) is constant (since $u(X(t), t)$ is constant)

Thus the characteristics are also here **straight lines**



## Nonlinear conservation laws

- Discontinuities can both appear and disappear in the solution to nonlinear conservation laws
- **Shock** (sv. *stötar*): discontinuities in the solution appearing when characteristics intersect
- **Rarefaction wave** (sv. *expansionsvåg*): a discontinuity is smeared out by diverging characteristics
- Thus, an important feature of numerical schemes for conservation laws: the ability to handle discontinuities in the solution!



## Finite volume methods for 1D conservation laws

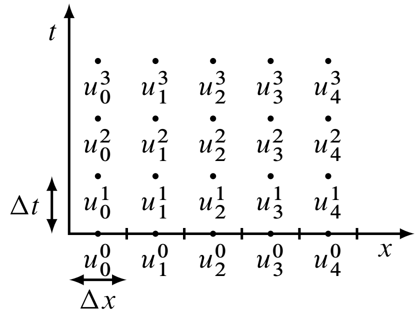
The finite volume method seeks approximations to **cell averages** at times $t_n$:

$$u_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n)\, dx$$

A family of **conservative**, **explicit** schemes:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) \tag{4}$$

$F_{i+1/2}^n \approx f(u(x_{i+1/2}, t_n))$: **numerical flux function**. Defines the particular method.



## The upwind method

Recall: $u_t + f(u)_x = u_t + f'(u)u_x = 0$. Thus:

$f'(u) > 0$: transport to the **right**
$f'(u) < 0$: transport to the **left**

Motivates the choice

$$F_{i+1/2} = \begin{cases} f(u_i^n) & \text{if } f'(u) > 0, \\ f(u_{i+1}^n) & \text{if } f'(u) < 0 \end{cases}$$

The flux function is evaluated in the "upwind direction."

Yields the scheme:

$$u_i^{n+1} = \begin{cases} u_i^n - \frac{\Delta t}{\Delta x}\left(f(u_i^n) - f(u_{i-1}^n)\right) & \text{if } f' > 0, \\ u_i^n - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^n) - f(u_i^n)\right) & \text{if } f' < 0 \end{cases}$$



## The Lax–Friedrich scheme

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) + \frac{\Delta t}{2\Delta x}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right]$$

Belongs to the family (4) of schemes with the numerical flux

$$F_{i+1/2}^n = \frac{1}{2}\left[f(u_{i+1}^n) + f(u_i^n)\right] - \frac{\Delta x}{2\Delta t}\left[u_{i+1}^n - u_i^n\right]$$

The upwind and Lax–Friedrich schemes behave similarly:

- Very robust and stable
- Only first-order accurate in space and time for smooth solutions: need very small $\Delta t$, $\Delta x$ for accurate solutions
- Tend to smear out sharp spatial gradients



## Second-order-accurate methods

In lab 1 we tested a method that was second order in space and time: the Richtmyer two-step Lax–Wendroff method

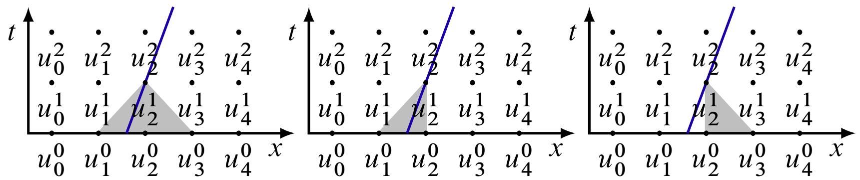
- Performs **much** better for smooth solutions
- However, tends to generate oscillations around discontinuities

More advanced methods: "high-resolution methods"

- Second-order accurate (or better) in smooth regions of the solution
- A *limiter* or *artificial dissipation* used in the cells around discontinuities to avoid oscillations. The scheme typically reduces to an upwind-like scheme around the shock
- The scheme becomes nonlinear! Needs "sensors" that detects regions of sharp gradients.



## The CFL condition




Lax–Friedrich
Lax–Wendroff

Correct upwind
(Stable)

Incorrect upwind
(unstable)

A necessary condition for stability:

The characteristics through the "update" point must pass through the **numerical domain of dependency** (the gray region)

From picture, for schemes involving points $u_{i-1}^n$, $u_i^n$, and $u_{i+1}^n$,

$$\frac{\Delta t}{\Delta x} \le \frac{1}{c}$$

Thus, it is necessary that $\Delta t \le \Delta x / c$