Review and some extensions: FEM

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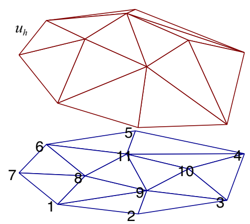
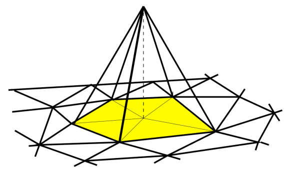
10 oktober 2011

Approximations and representations of functions in the Finite Element Method



- The given domain is **triangulated**
- Functions u_h are glued together from simple functions, typically polynomials, defined on each element of the triangulation
- Easiest example: u_h is **continuous** and **piecewise linear**:
  - The nodal values $\mathbf{u} = (u_1, \dots, u_n)$ are stored
  - The function can be recreated from the nodal values by using the "hat" basis functions $\phi_i(\boldsymbol{x})$, $i = 1, \dots, n$



$$u_h(\boldsymbol{x}) = \sum_{i=1}^{n} u_i \phi_i(\boldsymbol{x})$$

Model problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= g && \text{on } \Gamma_N \end{aligned} \quad \text{Finite-element approximation} \quad \Rightarrow \quad \mathbf{Au} = \mathbf{b}$$

Steps in the process:

1. Apply a **test function** v to the PDE and integrate. The test function is arbitrary but **vanishing** at the part of the boundary where u is known (Γ_D)
2. Use **integration by parts** to move one derivative from the **trial function** (u) to the test function. Avoids the need to differentiate the trial function twice. Yields a **variational problem**.
3. Apply a **Galerkin approximation** of the variational form using continuous, piecewise-linear functions

"Test vectors" for linear systems

Test functions are nothing mysterious. The idea can also be applied to linear systems:

$$\mathbf{Au} = \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{v}^T\mathbf{Au} = \mathbf{v}^T\mathbf{b} \qquad \forall \mathbf{v} \in \mathbb{R}^n$$

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 &= b_1 \\ a_{21}u_1 + a_{22}u_2 &= b_2 \end{aligned}$$

$$\Longleftrightarrow$$

$$v_1(a_{11}u_1 + a_{12}u_2) + v_2(a_{21}u_1 + a_{22}u_2) = v_1 b_1 + v_2 b_2 \quad \forall v_1, v_2$$

- A system of n linear equation is equivalent to **one** equation—a variational form that contains an **arbitrary** "test vector" $\mathbf{v}$
- The variational form is equivalent to the original problem
- By choosing $\mathbf{v} = \mathbf{e}_i$, $i = 1, \dots, n$ ($\mathbf{e}_i$ = standard basis vectors), we recover the original system of equations $\mathbf{Au} = \mathbf{b}$.

Variational form derivation

The boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega \tag{1a}$$
$$u = 0 \quad \text{on } \Gamma_D \tag{1b}$$
$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_N \tag{1c}$$

Let v be an arbitrary smooth test function with $v(\boldsymbol{x}) = 0$ on Γ_D (where u i known). Multiply (1a) with v and integrate, using Green's formula:

$$\int_\Omega vf\, dV = -\int_\Omega v\Delta u\, dV$$
$$= -\int_{\Gamma_D} \underbrace{v}_{=0} \frac{\partial u}{\partial n}\, dS - \int_{\Gamma_N} v \underbrace{\frac{\partial u}{\partial n}}_{=g}\, dS + \int_\Omega \nabla v \cdot \nabla u\, dV$$
$$= -\int_{\Gamma_N} vg + \int_\Gamma \nabla v \cdot \nabla u\, dS$$

The variational form

Theorem

If u solves the Poisson problem (1), then

$$\int_\Omega \nabla v \cdot \nabla u\, dV = \int_{\Gamma_N} vg\, dS + \int_\Omega vf\, dV$$

for each smooth function v vanishing on $\partial\Omega$.

However, the variational form is meaningful **even without reference to** (1).

Introduce the *energy space* (a *Sobolev space* of order one)

$$V = \left\{ v \mid \int_\Omega |\nabla \boldsymbol{v}|^2\, dV < +\infty \ \text{ and } v = 0 \text{ on } \Gamma_D \right\}$$

$V \subset H^1(\Omega)$

Weak solutions

The variational problem:

Find $u \in V$ such that

$$\int_\Omega \nabla v \cdot \nabla u\, dV = \int_{\Gamma_N} vg\, dS + \int_\Omega vf\, dV \qquad \forall v \in V \tag{2}$$

- Solutions to variational problem (2) are called *weak solutions* to boundary-value problem (1).
- Theorem on previous page: solutions to boundary-value problem (1) are weak solutions
- Weak solutions are also solutions boundary-value problem (1) provided that f, g, and Ω are regular enough

Finite element approximations

A *Galerkin approximation* to u is obtain by choosing $V_h \subset V$ and solving

Find $u_h \in V_h$ such that

$$\int_\Omega \nabla v_h \cdot \nabla u_h\, dV = \int_{\Gamma_N} v_h g\, dS + \int_\Omega v_h f\, dV \qquad \forall v_h \in V_h \tag{3}$$

Choosing V_h to be **continuous functions** that are **polynomials** on each **element** in a **triangulation** of Ω, we obtain a *Finite-Element Approximation* of boundary-value problem (1).

The algebraic problem

Substitute

$$u_h = \sum_{j=1}^{N} u_j \phi_j$$

into the finite-element problem (3):

$$\sum_{j=1}^{N} u_j \int_\Omega \nabla v_h \cdot \nabla \phi_j \, dV = \int_{\Gamma_N} v_h g \, dS + \int_\Omega v_h f \, dV \quad \forall v_h \in V_h$$

In particular, since $\phi_i \in V_h$

$$\sum_{j=1}^{N} u_j \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, dV = \int_{\Gamma_N} \phi_i g \, dS + \int_\Omega \phi_i f \, dV$$

for $i = 1, \ldots, N$. This is a linear system in the coefficients u_j:

$$\mathbf{Au} = \mathbf{b}$$

Stiffness matrix properties

- Matrix $\mathbf{A}$ is *sparse*. Most elements are zero. Generally true for all matrices from finite-element discretizations!
- In this case:
  - $\mathbf{A}$ is *symmetric* ($\mathbf{A}^T = \mathbf{A}$)
  - $\mathbf{A}$ is *positive definite*, and the linear system thus has a unique solution

  These properties are due to properties of this particular boundary-value problems.

Boundary condition terminology

PDE problem	**Variational problem**
Dirichlet BC	Essential BC (constraints explicitly enforced in the definition of spaces)
Neumann and Robin BC	Natural BC (conditions included in the variational problem)