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Teknisk berÃd'kningsvetenskap II

Case Study II: Models of transport, waves, and shallow waters Review questions and exercises. Selected answers

1 Modeling

1.1 Review questions

1. Specify the conservative form, the primitive form, and the integral form of a scalar conservation law in one space dimension.

   Answer:

$$
\begin{aligned}
u_t + f(u)_x &= 0 \quad \text{(conservative)},\\
u_t + f(u)' u_x &= 0 \quad \text{(primitive)},\\
\frac{d}{dt}\int_a^b u\,dx + f(u(b,t)) - f(u(a,t)) &= 0 \quad \text{(integral)},
\end{aligned}
\tag{1}
$$

2. What is the significance of the characteristics associated with a conservation law?

   Answer:

   The characteristics to the conservation law $u_t + f(u)_x = 0$ are the lines $(X(t), t)$ in the x–t-plane such that smooth solutions are constant along each characteristic, i.e.

$$
\frac{d}{dt} u(X(t), t) = 0. \tag{2}
$$

3. Sketch the characteristics for Burgers equation associated with the initial condition u_0 in figure 1, and sketch qualitatively the initial development of the solution with this boundary condition. Note the different behaviors to the right and left of $x = 0$.

4. Under which condition is the momentum conserved in a fixed amount of fluid?

   Answer:

   Momentum is conserved if there are no external forces acting on the fluid.

1.2 Exercises

1. Define the characteristics of the conservation law $u_t + f(u)_x = 0$.

   Answer:

   The characteristics are the curves $(X(t), t)$ such that

$$
\begin{aligned}
&\dot{X}(t) = f'\big(u(X(t), t)\big) \qquad t > 0,\\
&X(0) = x_0.
\end{aligned}
\tag{3}
$$

   Since smooth solutions are constant along the characteristics, $u(X(t), t) = u(x_0, t)$, the equation can also be written $\dot{X}(t) = f'\big(u_0, t)\big)$.

2. Let $u(x, t)$ be a velocity field computed by solving a scalar conservation law in one space dimension; $u > 0$ corresponds to a velocity directed along increasing x coordinate. Write up the ODE satisfied by the x-coordinate $p(t)$ of a massless particle that starts at position x_0 and is transported by the velocity field along the x axis.

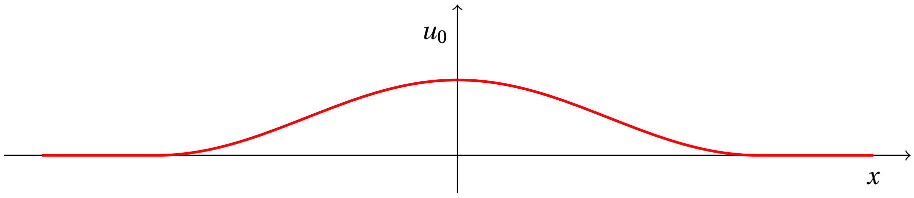

FIGURE 1. An initial condition

Answer:

$$\begin{aligned} p' &= u(p,t) \qquad t>0 \\ p(0) &= x_0 \end{aligned} \tag{4}$$

3. Rewrite following equations in conservative form and specify the flux function:
   (a)
   $$u_t + uu_x = 0 \tag{5}$$
   (b)
   $$u_t + uu_x + vu_y = 0, \tag{6}$$
   under the condition that $u_x + v_y = 0$.
   (c)
   $$uu_t + u^2u_x = 0, \tag{7}$$
   as a conservation law in u^2.

   Remark 1. Note that equation (7) is obtained by multiplying equation (5) by u. Equations (5) and (7) are thus equivalent for smooth solutions. However, their conservative forms will be different and corresponding shock propagation speeds will differ!

   Answer:
   (a) $u_t + \frac{1}{2}(u^2)_x = 0$; flux function: $f(u) = \frac{1}{2}u^2$
   (b)
   $$\begin{aligned} u_t + uu_x + vu_y &= u_t + (u^2)_x - u_xu + (vu)_y - v_yu \\ &= u_t + (u^2)_x + (vu)_y - (u_x + v_y)u \\ \left[\text{since } u_x + u_y = 0\right] &= u_t + (u^2)_x + (vu)_y = u_t + \nabla\cdot\boldsymbol{f}(u,v), \end{aligned} \tag{8}$$
   where
   $$\boldsymbol{f}(u,v) = (u^2, vu) \tag{9}$$
   (c) $(u^2)_t + \frac{2}{3}\left[(u^2)^{3/2}\right]_x = 0$; flux function: $f(u) = \frac{2}{3}(u^2)^{3/2}$

4. For arbitrary twice differentiable function f and g, show that
   $$p(x,t) = f(x+ct) + g(x-ct) \tag{10}$$
   is a solution to the wave equation
   $$p_{tt} - c^2p_{xx} = 0. \tag{11}$$
   From expression (10), conclude the form of the characteristics associated with the wave equation (11).

Answer:

The p of expression (10) satisfies the wave equation since $p_{tt} = c^2 f''(x+ct)$ and $p_{xx} = f''(x+ct)$. The characteristics associated with equation (11) are the lines $(x+ct, t)$ and $(x-ct, t)$ in the (x, t) plane. (Note that the second-order wave equation has *two* sets of characteristics!)

5. (a) Rewrite the wave equation (11) as a system of conservation laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0} \tag{12}$$

in $\boldsymbol{u} = (u, v)$, where $u = -cp_x$ and $v = p_t$, and specify the 2-by-2 matrix $\boldsymbol{A}$ in the representation $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ of the flux function.

(b) By defining variables w and z as linear combinations of the above u and v, rewrite equation (12) as two uncoupled scalar transport equations, representing transport in the positive and negative x-direction respectively. (These are called *characteristic variables*).

Answer:

(a)

$$u_t + cv_x = 0, \tag{13a}$$

$$v_t + cu_x = 0 \tag{13b}$$

or, in the vector form $\boldsymbol{u}_t + (\boldsymbol{A}\boldsymbol{u})_x = 0$ with

$$\boldsymbol{A} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \tag{14}$$

(b) Adding and subtracting equations (13a) and (13b) yields

$$\begin{aligned} (u+v)_t + c(u+v)_x &= 0, \\ (u-v)_t - c(u-v)_x &= 0. \end{aligned} \tag{15}$$

Thus, the variables $w_+ = u + v$ and $w_- = u - v$ represents transport in the positive and negative directions along the x-axis.

6. Consider Burgers equation with smooth initial data $u_0(x)$,

$$\begin{aligned} u_t + \frac{1}{2}(u^2)_x &= 0 \qquad t > 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{16}$$

Claim: The characteristics will cross (and thus a shock form) if $u'(x) < 0$ at any point x, and the time for the first occurrence of crossing characteristics is

$$T^* = -\frac{1}{\min_x u'(x)}. \tag{17}$$

(a) Compute T^* for the initial condition (figure 1)

$$u_0(x) = \begin{cases} 1 + \cos \pi x & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

(b) Prove the above claim.

Answer:

(a) $T^* = 1/\pi$.

(b) We consider two characteristic lines in the (x,t) plane emanating in points x_0 and x_1 where $x_0 < x_1$. For Burgers equation, the characteristics are in parametric form $(x_0 + u_0(x_0)t, t)$ and $(x_1 + u_0(x_1)t, t)$, where $u_0(x)$ is the given initial value. If the characteristic lines cross at a time T, then

$$x_0 + u_0(x_0)T = x_1 + u_0(x_1)T, \tag{19}$$

which by the mean-value theorem (using that u_0 is smooth) can be written

$$-(x_0 - x_1) = T(u(x_0) - u(x_1)) = Tu'(\xi)(x_0 - x_1) \tag{20}$$

for some $\xi \in (x_0, x_1)$. That is,

$$Tu'(\xi) = -1, \tag{21}$$

which will have a solution for $T > 0$ if $u'(\xi) < 0$, and the minimal time T^* will satisfy

$$T^* = -\frac{1}{\min_x u'(x)}. \tag{22}$$

7. Generalize to two space dimensions x, y, the derivation of the shallow water equations over a planar bottom at $z = 0$. In this case, the unknown functions will be the water level $h(x,y)$ and components $u(x,y)$, $v(x,y)$ of the horisontal velocity vector $\boldsymbol{u} = (u, v, 0)$. *Hint:* There will be two moment balance laws, one for ρu and one for ρv.

2 FVM

2.1 Review questions

1. The family of conservative, explicit finite-volume schemes associated with conservation law $u_t + f(u)_x = 0$ that we consider is

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) \tag{23}$$

   (a) What is approximated by the quantities u_i^n?

   (b) What is the numerical flux function $F_{i+1/2}$ for the upwind scheme when $f(u) = cu$ in the case (i) $c > 0$, and (ii) $c < 0$?

2. Give (a least) one good and one bad property with the upwind scheme, the Lax–Friedrich method, and the Richmyer two-step Lax–Wendroff method.

   Answer:

   (a) *Upwind scheme:* (+) robust also for discontinuous solutions; (-) diffusive, only first-order accurate.
   Lax–Friedrich scheme: similar properties as upwind. In addition (+): information of upwind direction not needed; (-): even more diffusive than the upwind scheme.
   The Richmyer two-step Lax–Wendroff method: (+): much more accurate (second-order in time and space) than the upwind and Lax–Friedrich methods for smooth solutions; (-): oscillations will occur around discontinuities in the solution.

3. What is meant by the CFL condition for a numerical scheme associated with a conservation law?

4. Are we guaranteed that a numerical scheme is stable if it satisfies the CFL condition? Can a numerical scheme violate the CFL condition and still be stable?

   Answer:

   No to both questions. The CFL condition is a necessary but not a sufficient condition for stability, and a numerical scheme that violates the CFL condition is unstable.

2.2 Exercises

1. Show that the upwind numerical flux for the particular case $f(u) = cu$ can be written

$$F^n_{i+1/2} = \frac{1}{2}\big(f(u^n_{i+1}) + f(u^n_i)\big) - \frac{|c|}{2}(u^n_{i+1} - u^n_i), \tag{24}$$

and that the update formula (23) with numerical flux (24) becomes

$$u^{n+1}_i = u^n_i - \frac{\Delta t}{2\Delta x}\left[f(u^n_{i+1}) - f(u^n_{i-1})\right] - \frac{|c|\Delta t}{2\Delta x}\left[2u^n_i - u^n_{i-1} - u^n_{i+1}\right]. \tag{25}$$

Remark 2. Expression $\big(f(u_{i+1}) - f(u_{i-1})\big)/(2\Delta x)$ constitutes a *central difference approximation* of $f(u)_x$, and $-(2u_i - u_{i-1} - u_{i+1})\Delta x^2$ a finite-difference approximation of $u''(x_i)$. Expression (25) can thus be viewed as a discretization of the equation

$$u_t + f'(u)_x = |c|\Delta x u''. \tag{26}$$

with a central difference approximation of $f(u)_x$. Hence, an *upwind* discretization of $u_t = f(u)_x$ for $f(u) = cu$ is equivalent to a *central* discretization of the conservation law with an added *artificial dissipation* (the last term in expression (25) and the right side of equation (26)). This interpretation is consistent the very diffusive properties of the upwind discretization.

2. A common implementation of the upwind scheme for nonlinear f is to use formula (24) and substitute c with an approximation of f' at the interface. Verify that we obtain the upwind scheme for Burgers equation when using the numerical flux (24) with $c = (u_i + u_{i+1})/2$.

3. Show that the Lax–Friedrich method for the particular case $f(u) = cu$ will be the same as the upwind method for a particular choice of time step Δt.

4. Show that the upwind scheme exactly solves the transport equation $u_t + cu_x = 0$ if the time step is chosen at the stability limit for the CFL condition. (This property does *not* hold for nonlinear conservation laws!)

Answer:

The upwind scheme for the transport equation:

$$u^{n+1}_i = \begin{cases} u^n_i - \dfrac{c\Delta t}{\Delta x}\left(u^n_i - u^n_{i-1}\right) & c > 0 \\ u^n_i - \dfrac{c\Delta t}{\Delta x}\left(u^n_{i+1} - u^n_i\right) & c < 0 \end{cases} \tag{27}$$

At the CFL stability limit

$$\frac{c\Delta t}{\Delta x} = \pm 1, \tag{28}$$

update (27) simply becomes

$$u^{n+1}_i = \begin{cases} u^n_{i-1} & c > 0, \\ u^n_{i+1} & c < 0. \end{cases} \tag{29}$$

The exact solution $u(x, t)$ to the transport equation is constant along the characteristic lines, so for each x and t,

$$u(x, t_n) = u(x + ct, t_n + t) \tag{30}$$

First consider the case $c > 0$. Choosing $t = \Delta t$ and $x = x_{i-1}$ in (30), we find that the exact solution satisfies

$$u(x_{i-1}, t_n) = u(x_{i-1} + c\Delta t, t_{n+1}) = \big[\text{when (28) holds}\big] = u(x_i, t_{n+1}). \tag{31}$$

Likewise, when $c < 0$, choosing $t = \Delta t$ and $x = x_{i+1}$ in (30) yields that

$$u(x_{i+1}, t_n) = u(x_{i+1} + c\Delta t, t_{n+1}) = \big[\text{when (28) holds}\big] = u(x_i, t_{n+1}). \tag{32}$$

Thus, the exact solution in the cell centers is transported exactly like the numerical solution, so if the numerical solution is initiated with exact data, it will be exact at the cell centers at each time step.