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Teknisk beräkningsvetenskap II

Case Study I: Groundwater flow modeling using FEM. Review questions and exercises. Selected answers

1 Modeling

1.1 Review questions

1. In general, the fluid flowing in a porous medium is affected by *adhesive*, *capillary*, *inertial* (sv. *tröghetskrafter*), and *gravitational* forces.
   (a) Explain the causes of adhesive, capillary, and inertial forces.
   (b) What forces are usually dominating in a porous medium?
   (c) Which force vanishes in a saturated medium? Why?

   Answer:
   (a) *Adhesive* forces: the force at fluid–solid interfaces caused by the fluid's inner friction (viscosity); *capillary* forces: cased by surface tension; *inertial* forces: associated with acceleration of the fluid.
   (b) Adhesive and capillary.
   (c) The capillary forces vanish in a saturated medium because the voids are completely filled with fluid.

2. Why would we want to use continuum approximations of porous media?

   Answer:
   To simulate the large-scale (macro scale) effects in i porous media using an exact, micro description of the medium is computationally too demanding, and the exact microstructure is usually unknown and unimportant for the macro-scale effects.

3. For a porous medium, define the *porosity* and explain the concept of a *Representative Elementary Volume*.

   Answer:
   Porosity is the volume of the void space divided by measure of the volume under consideration. A representative elementary volume is the smallest volume for which an averaged property (such as porosity) acquires a well-defined stable value.

4. Explain the meaning of the terms *homogeneous*, *heterogeneous*, *isotropic*, and *anisotropic*.

   Answer:
   Homogeneous: same at each point in space; *heterogeneous:* a property that varies in space (also called *inhomogeneous*); *isotropic:* a property that is the same in different directions; *anisotropic:* a property that is different in different directions.

5. What is meant by the *apparent velocity* (or Darcy velocity) $\boldsymbol{u}$ of a fluid in a porous medium.

   Answer:
   Let V be an arbitrary control volume that is much larger than the representative volume element. The apparent velocity is an averaged velocity field that causes the same volume flux through the surface of V as the exact velocity field.

6. Derive the law of mass conservation,

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{1}$$

   or the flow of a fluid with density ρ and apparent velocity $\boldsymbol{u}$ in a saturated porous medium with porosity ϕ. Also discuss the case when the density is constant.

Answer:

See the lecture notes or the provided material (Bastian).

7. Does Darcy's law hold for all porous media?

Answer:

No, the medium has to be saturated, and a representative elementary volume has to be possible to define.

1.2 Exercises

1. If the apparent velocity field satisfies $\boldsymbol{u} = -\kappa \nabla h$ and if the hydraulic conductivity κ is a constant, show that $\boldsymbol{u}$ is irrotational.

Answer:

$\nabla \times \boldsymbol{u} = -\nabla \times \kappa \nabla h = -\kappa \nabla \times \nabla h = 0.$

2. The velocity field

$$\boldsymbol{u} = \frac{a}{x^2 + y^2}(x, y) \tag{2}$$

represents a *point source* ($a > 0$) or a *point sink* ($a < 0$) located at the origin. In groundwater flow, a water well (sv. *brunn*) can be modeled as a point sink.

(a) Write velocity field (2) in polar coordinates.

(b) What is the pressure head associated with velocity field (2)?

(c) Show that $\nabla \cdot \boldsymbol{u} = 0$ everywhere except at the origin.

(d) Calculate the flux (in m^2/s) through a circle with radius $R > 0$ centered at the origin. How should a be chosen in order for the velocity field to represent a well where 1 m^2/s is pumped out at the origin?

(e) Show that flux of $\boldsymbol{u}$ (in m^2/s) through an arbitrary closed surface S that does not include the origin is zero.

(f) Calculate the flux of $\boldsymbol{u}$ through an arbitrary closed surface that includes the origin. Interpret the results of (d) and (e) physically.

Answer:

(a) $\boldsymbol{u} = \dfrac{a}{r}\hat{\boldsymbol{r}}$

(b) $h = a \log r + C$

(c) In polar coordinates: for $r > 0$,

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{a}{r}) = 0. \tag{3}$$

In Cartesian coordinates: for $(x, y) \neq (0, 0)$

$$\nabla \cdot \boldsymbol{u} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} + \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = 0. \tag{4}$$

(d) S: circle of radius R centered around the origin.

$$\int_S \boldsymbol{n} \cdot \boldsymbol{u}\, dS = \left[\text{Polar coordinates, expr. (3)}\right] = \int_0^{2\pi} a\, d\theta = 2\pi a. \tag{5}$$

Note that the flux is independent of the sphere's radius! Thus, $a = 1/(2\pi)$ will give a flux of unit strength.

(e) S: closed surface that does not enclose the origin; V: the volume interior to S.

$$\int_S \boldsymbol{n} \cdot \boldsymbol{u}\, dS = \int_V \nabla \cdot \boldsymbol{u}\, dV = 0, \tag{6}$$

since $\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \neq 0$.

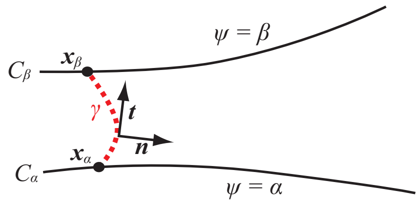

FIGURE 1. The solid curves markes streamlines C_α and C_β, where $\psi = \alpha$ and $\psi = \beta$, respectively. The dashed curve, with normal and tangent vectors $\boldsymbol{n}$ and $\boldsymbol{t}$, starts at $\boldsymbol{x}_\alpha \in C_\alpha$ and ends at $\boldsymbol{x}_\beta \in C_\beta$.

(f) S: a closed surface enclosing the origin; V: the volume whose boundary is S. Let B_ϵ be a ball with radius ϵ, fitted inside V ($B_\epsilon \subset V$) and centered around the origin, and let S_ϵ be the sphere that constitutes the boundary of B_ϵ. Then,

$$\begin{aligned}\int_S \boldsymbol{n}\cdot\boldsymbol{u}\,dS &= \int_S \boldsymbol{n}\cdot\boldsymbol{u}\,dS - \int_{S_\epsilon} \boldsymbol{n}\cdot\boldsymbol{u}\,dS + \int_{S_\epsilon} \boldsymbol{n}\cdot\boldsymbol{u}\,dS \\ [\text{Gauss' thm}] &= \underbrace{\int_{V\setminus B_\epsilon} \nabla\cdot\boldsymbol{u}\,dV}_{=0 \text{ by } (6)} + \int_{S_\epsilon} \boldsymbol{n}\cdot\boldsymbol{u}\,dS = \int_{S_\epsilon} \boldsymbol{n}\cdot\boldsymbol{u}\,dS = [(5)] = 2\pi a. \end{aligned} \tag{7}$$

By mass conservation, the flux through an arbitrary surface enclosing the origin will be the same, since the sole mass source is the point source at the origin.

3. In the labs, you plotted *streamlines*. The following sequence of problems outlines some of the mathematics associated with streamlines in two space dimensions, for which the concept of *stream function* can be used.

(a) A function ψ that satisfies

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{8}$$

is called a *stream function* associated with the velocity field $\boldsymbol{u} = (u, v)$. Show that $\nabla\cdot\boldsymbol{u} = 0$ if condition (8) holds.

(b) Derive an equation that should be satisfied for the stream function in order for associated velocity field to be irrotational.

(c) A *level curve* of a stream function is a curve for which ψ is constant, that is, the set of points C_α for which $\psi = \alpha$. Show that the level curves of a stream functions are parallel with the velocity field.

Hint: The directional derivative of ψ in the direction of the velocity field is $(\boldsymbol{u}\cdot\nabla)\psi$.

(d) Let C_α and C_β be two level curves of the stream function. Thus $\psi = \alpha$ and $\psi = \beta$ at all points on C_α and C_β, respectively. Moreover, let γ be an arbitrary curve that starts at a point $\boldsymbol{x}_\alpha \in C_\alpha$ and ends at a point $\boldsymbol{x}_\beta \in C_\beta$ (Figure 1). The flux of the velocity $\boldsymbol{u}$ across γ is

$$Q = \int_\gamma \boldsymbol{n}\cdot\boldsymbol{u}\,ds, \tag{9}$$

where $\boldsymbol{n}$ is a normal vector field to γ. (That is, Q m^2/s of fluid is passing through γ if $\boldsymbol{u}$ is given in m/s). Show that $Q = \beta - \alpha$.

Hints: (i) Parameterize the curve γ with a parameter s such that for $s \in [0, 1]$, $\boldsymbol{x}(s) \in \gamma$ and such that $\boldsymbol{x}(0) = \boldsymbol{x}_\alpha$ and $\boldsymbol{x}(1) = \boldsymbol{x}_\beta$. By the fundamental theorem of integral calculus,

$$\beta - \alpha = \psi(\boldsymbol{x}_\beta) - \psi(\boldsymbol{x}_\alpha). = \int_0^1 \frac{d}{ds}\psi(\boldsymbol{x}(s))\,ds. \tag{10}$$

(ii) Use the chain rule of differentiation for the integrand in expression (10), and use the fact that $\boldsymbol{t} = (-n_y, n_x)$ is the tangent vector to γ illustrated in figure (1) (n_x and n_y are the x- and y-coordinates of the normal vector $\boldsymbol{n}$).

2 FEM

2.1 Review questions

1. What is meant by a weak solution to a boundary-value problem for Poisson's equation?

   Answer:

   The solution to the variational problem constitutes a weak solution to the boundary value problem for Poisson's equation.

2. Give two reasons to use integration by parts (Green's first identity) when deriving the variational form for the Poisson problem that constitutes the basis for finite-element discretization.

   Answer:

   (a) The finite-element functions are typically once, but not twice, differentiable, and the variational problem derived using integration by parts contains only first derivatives, as opposed to the partial differential equation.
   (b) The use of Green's first identity makes it possible to include Neumann and Robin boundary conditions in the variational formulation.

3. Explain the difference between *essential* and *natural* boundary condition for a variational problem.

   Answer:

   An essential boundary condition is a condition that is explicitly enforced on the functions in the definition of the spaces. A natural boundary condition is not explicitly specified but is included in the variational formulation.

2.2 Exercises

1. For following boundary-value problems, derive variational forms, define a FE approximation using continuous, piecewise-linear functions on a uniform mesh, and specify the linear system associated with the FE approximation.

   (a)
   $$\begin{aligned} -u'' &= f \qquad \text{in } (0,1), \\ u(0) &= 0, \\ u'(1) &= 0. \end{aligned}$$

   (b)
   $$\begin{aligned} -u'' &= f \qquad \text{in } (0,1), \\ u(0) &= g, \\ u(1) &= 0. \end{aligned}$$

   Hint: Letting $\phi_i(x)$ denote the standard "hat" basis function centered at point $x_i = ih$, $i = 0, \ldots, N$, $h = 1/N$, write the finite element solution as

   $$u_h(x) = \sum_{i=0}^{N} u_i\phi_i(x) = g\phi_0(x) + \underbrace{\sum_{i=1}^{N} u_i\phi_i(x)}_{\hat{u}_h(x)} = g\phi_0(x) + \hat{u}_h(x),$$

   where the second equality follows from the boundary condition at $x = 0$. Substitute the above expression into the variational form and move the term associated with $g\phi_0(x)$ to the right hand side (since it is a known "forcing"-type quantity). Then you will obtaine a linear system for the unknown coefficients in an expansion of $\hat{u}_h(x)$. Note that the function is *known* at the endpoints and *unknown* only in the mesh nodes in the strict interior interval. The order of the stiffness matrix should therefore be equal to the number of strict interior nodes.

(c)

$$\begin{aligned} -u'' + au' &= f \qquad &\text{in } (0,1),\\ u(0) = u(1) &= 0. \end{aligned}$$

(d)

$$\begin{aligned} -u'' + u &= f \qquad &\text{in } (0,1),\\ u'(0) = u'(1) &= 0. \end{aligned} \tag{11}$$

(e)

$$\begin{aligned} -(c(x)u')' &= f \qquad &\text{in } (0,1),\\ u(0) = u(1) &= 0, \end{aligned}$$

where $c(x) > 0$ on $[0,1]$. When computing the stiffness matrix, assume that $c(x)$ is piecewise constant in each element, so that $c_{i+1/2}$ is the value in interval (x_i, x_{i+1}).

Answer:

All problems (a)–(e) give systems of equation $\mathbf{Au} = \mathbf{b}$ with vectors and matrices as below. For all problems we use I intervals of length Δx. Note, however, that the order of matrices and vectors differ in problems (a)–(e) depending on the boundary conditions at the end points! The hat basis function associated with node i is called ϕ_i.

(a) Mesh points: $x_i = i\Delta x$, $i = 1, \dots, I$.

$$\mathbf{A} = \frac{1}{\Delta x}\begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \tag{12}$$

$$\mathbf{u} = (u_1, \dots, u_I)^T, \qquad b_i = \int_0^1 \phi_i f\,dx$$

(b) Mesh points: $x_i = i\Delta x$, $i = 1, \dots, I-1$.

$$\mathbf{A} = \frac{1}{\Delta x}\begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{u} = (u_1, \dots, u_{I-1})^T, \tag{13}$$

$$b_1 = \frac{g}{h} + \int_0^1 \phi_1 f\,dx, \qquad b_i = \int_0^1 \phi_i f\,dx \qquad \text{for } i = 2, \dots, I-1.$$

(c) Mesh points: $x_i = i\Delta x$, $i = 1, \dots, I-1$; $\mathbf{A} = \mathbf{K} + \mathbf{C}$ with

$$\mathbf{K} = \frac{1}{\Delta x}\begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{C} = \frac{a}{2}\begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ -1 & 0 & 1 & 0 & \cdots & \\ 0 & -1 & 0 & 1 & \cdots & \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{14}$$

$$\mathbf{u} = (u_1, \dots, u_{I-1})^T, \qquad b_i = \int_0^1 \phi_i f\,dx \qquad \text{for } i = 1, \dots, I-1.$$

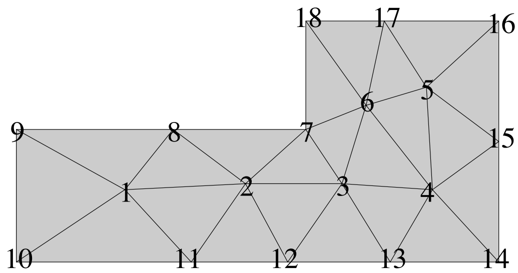

FIGURE 2. The domain for problem (17) and the triangulation.

(d) Mesh points: $x_i = i\Delta x$, $i = 0, \ldots, I$; $\mathbf{A} = \mathbf{K} + \mathbf{M}$ with

$$\mathbf{K} = \frac{1}{\Delta x}\begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{M} = \frac{h}{6}\begin{pmatrix} 2 & 1 & 0 & \cdots & \\ 1 & 4 & 1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \tag{15}$$

$$\mathbf{u} = (u_0, \ldots, u_I)^T, \qquad b_i = \int_0^1 \phi_i f \, dx \qquad \text{for } i = 0, \ldots, I.$$

(e) Mesh points: $x_i = i\Delta x$, $i = 1, \ldots, I-1$;

$$\mathbf{A} = \frac{1}{\Delta x}\begin{pmatrix} c_{1/2}+c_{3/2} & -c_{3/2} & 0 & \cdots & \\ -c_{3/2} & c_{3/2}+c_{5/2} & -c_{5/2} & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & -c_{I-5/2} & c_{I-5/2}+c_{I-3/2} & -c_{I-3/2} \\ 0 & 0 & 0 & -c_{I-3/2} & c_{I-3/2}+c_{I-1/2} \end{pmatrix},$$

$$\mathbf{u} = (u_1, \ldots, u_{I-1})^T, \qquad b_i = \int_0^1 \phi_i f \, dx \qquad \text{for } i = 1, \ldots, I-1. \tag{16}$$

2. Boundary-value problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{17}$$

where Ω is the domain marked gray in figure 2 and $\partial\Omega$ its boundary, is numerically solved with the FEM using continuous, piecewise-linear functions on the triangulation marked in the figure. The discretization yields a system of equations for the unknown node values.

(a) What order has the matrix of this linear system (the stiffness matrix)?

(b) Sketch the sparsity pattern of the stiffness matrix by marking with 0 the elements that are necessarily zero and with × the rest of he elements.

Answer:

(a) The stiffness matrix is of order 6 (the number of inner nodes).

(b) Sparsity pattern of the stiffness matrix:

$$\begin{pmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & 0 & \times & \times \end{pmatrix} \tag{18}$$

3. Consider the following boundary-value problem for the Helmholtz equation

$$\begin{aligned} -\Delta p + k^2 p &= 0 \qquad && \text{in } \Omega, \\ \frac{\partial p}{\partial n} &= 0 && \text{on } \Gamma_w. \\ \mathrm{i}kp + \frac{\partial p}{\partial n} &= 2\mathrm{i}kg && \text{on } \Gamma_s, \end{aligned} \tag{19}$$

where $k > 0$ and where $\mathrm{i} = \sqrt{-1}$. Derive a variational form for problem (19).

Remark 1. Problem (19) models the complex pressure amplitude p associated with single-frequency acoustic wave propagation in a room Ω. The pressure as a function of time will be the real part of $P(\boldsymbol{x}, t) = p(\boldsymbol{x})e^{\mathrm{i}\omega t}$, where ω is the angular frequency of the wave. The *wave number* is $k = \omega/c$, where c is the speed of sound. The walls of the room (the boundary of Ω) are sound hard (solid) at Γ_w. The boundary condition at Γ_s models an opening towards a ventilation duct, through which a wave with amplitude g enters the room, and through which sound also can escape.

Answer:

Function space: $H^1(\Omega)$ (the function as well as all partial derivatives are square integrable).
Variational problem: find $p \in H^1(\Omega)$ such that

$$\int_\Omega \nabla q \cdot \nabla p \, dV - k^2 \int_\Omega qp \, dV + \mathrm{i}k \int_{\Gamma_s} qp \, dS = 2\mathrm{i}k \int_{\Gamma_s} qg \, dS \qquad \forall q \in H^1(\Omega). \tag{20}$$

4. Small transversal[1] displacements $u(x)$ of a transversally loaded elastic cantilever[2] beam of unit length can be modelled by the classic Euler–Bernoulli beam equation

$$\begin{aligned} (D(x)u'')'' &= f \quad \text{in } (0,1), \\ u(0) &= 0, \\ u'(0) &= 0, \qquad\qquad , \\ u''(1) &= 0, \\ (D(1)u''(1))' &= 0 \end{aligned} \tag{21}$$

where $D(x) \geq \alpha > 0$, for each $x \in [0,1]$; the function D specifies the material and geometric properties of the beam (the product of Young's modulus for the material and the moment of inertia for the cross section of the beam).

Derive a variational form of equation (21) with equal number of derivatives for the trial and test functions. Also specify a suitable energy space to define weak solutions. Suggest a suitable space of finite-element functions. (Continuous, piecewise-linear functions is *not* a good choice. Why?)

[1] perpendicular to the beam's extension
[2] clamped in one end, free in the other

Answer:

Multiplying with a smooth test function v, and integrating by part *twice* yields the variational form

$$\int_0^1 D v'' u'' \, dx = \int_0^1 v f \, dx \tag{22}$$

Energy space:

$$V = \left\{ v \mid \int_0^1 (v'')^2 \, dx < +\infty, v(0) = 0, v'(0) = 0 \right\}. \tag{23}$$

Standard elements for 2nd-order problems (continuous, piecewise polynomials) is only once differentiable. (The derivative has a jump discontinuity at the nodes). A Galerkin scheme ($V_h \subset V$), will require finite element functions that are *twice* differentiable, that is, C^1 functions that are piecewise polynomials. A standard choice is cubic splines; the function and its derivative at the nodes are the degrees of freedom. Also called cubic Hermite elements.

5. Besides Green's first identity,

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS = \int_\Omega \nabla v \cdot \nabla u \, dV + \int_\Omega v \Delta u \, dV. \tag{24}$$

there are other integration-by-parts formulas that are useful when deriving variational forms associated with boundary-value problems. Show the following ones.

(a)

$$\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{U} v u \, dS = \int_\Omega v \nabla \cdot (\boldsymbol{U} u) \, dV + \int_\Omega (\boldsymbol{U} \cdot \nabla v) u \, dV, \tag{25}$$

where v, u are scalar-valued functions, and $\boldsymbol{U} = (U_1, \ldots, U_d)$ is vector valued.

Hint: Start with the product rule for $\nabla \cdot (\boldsymbol{U} v u)$.

(b)

$$\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v} \nabla \cdot \boldsymbol{u} \, dS = \int_\Omega (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{u}) \, dV + \int_\Omega (\boldsymbol{v} \cdot \nabla) \nabla \cdot \boldsymbol{u} \, dV \tag{26}$$

where $\boldsymbol{u}$, $\boldsymbol{v}$ are vector-valued functions.

Answer:

(a) The product rule:

$$\frac{\partial (U_i v u)}{\partial x_i} = v \frac{\partial (U_i u)}{\partial x_i} + U_i u \frac{\partial v}{\partial x_i}. \tag{27}$$

Summing for $i = 1, \ldots, d$, yields, in vector notation

$$\nabla \cdot (\boldsymbol{U} v u) = v \nabla \cdot (\boldsymbol{U} u) + u \boldsymbol{U} \cdot \nabla v \tag{28}$$

Integration and Gauss' theorem yields formula (25).

(b) Let

$$z = \nabla \cdot \boldsymbol{u}. \tag{29}$$

The product rule:

$$\frac{\partial (v_i z)}{\partial x_i} = \frac{\partial v_i}{\partial x_i} z + v_i \frac{\partial z}{\partial x_i}. \tag{30}$$

Summing over i, integrating, and using Gauss' theorem yields, in vector notation

$$\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v} z \, dS = \int_\Omega \nabla \cdot \boldsymbol{v} z \, dV + \int_\Omega (\boldsymbol{v} \cdot \nabla) z \, dV. \tag{31}$$

Substituting (29) yields requested expression.

6. We will derive variational formulations for steady so-called *advection–diffusion problems*,

$$-\nu\Delta u + \nabla\cdot(\boldsymbol{U}u) = 0, \tag{32}$$

with various boundary conditions. Integration by parts should always be used for the first term in equation (32). However, the second term involves only first derivatives, so integration by parts is not always needed. Whether or not integration by parts on the second term should be used will depend on the boundary conditions.

Remark 2. The two terms in equation (32) signifies processes of *diffusion* and *advection* (that is, transport), respectively. For instance, if $\boldsymbol{U}$ is the apparent velocity field in a saturated porous medium, equation (32) is a model for for the diffusion and transport of a pollutant. The variable u is then the concentration of the pollutant, and parameter ν the molecular diffusion constant.

(a) Derive a variational formulation of the following boundary-value problem:

$$\begin{aligned} -\nu\Delta u + \nabla\cdot(\boldsymbol{U}u) &= 0 \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{33}$$

Here, integrate by part only the first term in the PDE.

(b) Derive a variational formulation of the following boundary-value problem:

$$\begin{aligned} -\nu\Delta u + \nabla\cdot(\boldsymbol{U}u) &= 0 \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega \text{ whenever } \boldsymbol{n}\cdot\boldsymbol{U} < 0,\\ \nu\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \text{ whenever } \boldsymbol{n}\cdot\boldsymbol{U} = 0,\\ \nu\frac{\partial u}{\partial n} - \boldsymbol{n}\cdot\boldsymbol{U}u &= 0 \quad \text{on } \partial\Omega \text{ whenever } \boldsymbol{n}\cdot\boldsymbol{U} > 0, \end{aligned} \tag{34}$$

Here, use integration-by-parts formula (25) on the second term.

Remark 3. Since $\boldsymbol{n}$ is the outward-directed unit normal, condition $\boldsymbol{n}\cdot\boldsymbol{U} < 0$ signifies *inflow*, $\boldsymbol{n}\cdot\boldsymbol{U} > 0$ *outflow*, and $\boldsymbol{n}\cdot\boldsymbol{U} = 0$ an *impervious* boundary. Thus, in problem (34), a concentration g is specified at the inflow boundary, the concentration does not change across an impervious boundary. The condition on the outflow boundary models the situation when the pollutant is transported out of the domain.

Answer:

(a) Variational expression:

$$\nu\int_\Omega \nabla v\cdot\nabla u\, dV + \int_\Omega v\nabla\cdot(\boldsymbol{U}u)\, dV = 0 \tag{35}$$

Energy space for trial function u:

$$V_g = \left\{ v \mid \int_\Omega |\nabla v|^2\, dV < +\infty, v = g \text{ on } \partial\Omega \right\}; \tag{36}$$

for test functions v:

$$V = \left\{ v \mid \int_\Omega |\nabla v|^2\, dV < +\infty, v = 0 \text{ on } \partial\Omega \right\}; \tag{37}$$

(b) Variational form:

$$\nu\int_\Omega \nabla v\cdot\nabla u\, dV - \int_\Omega u(\boldsymbol{U}\cdot\nabla)v\, dV = 0. \tag{38}$$

Energy space for trial function u:

$$V_g = \left\{ v \mid \int_\Omega |\nabla v|^2\, dV < +\infty, v = g \text{ on } \partial\Omega \text{ whenever } \boldsymbol{n}\cdot\boldsymbol{U} < 0 \right\}; \tag{39}$$

for test functions v:

$$V_g = \left\{ v \mid \int_\Omega |\nabla v|^2 \, dV < +\infty, v = 0 \text{ on } \partial\Omega \text{ whenever } \boldsymbol{n} \cdot \boldsymbol{U} < 0 \right\}. \tag{40}$$

7. Use integration-by-parts formula (26) to derive a variational form for the following boundary-value problem:

$$\begin{aligned} \boldsymbol{u} - \nabla\nabla \cdot \boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{41}$$

(Equations of the above type appears for instance in acoustics).

Answer:

Variational form:

$$\int_\Omega \boldsymbol{v} \cdot \boldsymbol{u} \, dV + \int_\Omega \nabla \cdot \boldsymbol{v} \nabla \cdot \boldsymbol{u} \, dV = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \, dV \tag{42}$$

Remark 4. Although not asked for, it may be interesting to see the proper space for both the trial and test functions, which is a so-called H_{div} space:

$$V = \left\{ \boldsymbol{v} \mid \int_\Omega \left(\boldsymbol{v} \cdot \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})^2 \right) dV < +\infty, \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \partial\Omega \right\}. \tag{43}$$