Institutionen för datavetenskap
Umeå universitet

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Numeriska metoder för civilingenjörer

Theme: Rocket launches and ODEs
Part I

Study the theory (the lecture notes and relevant sections in the book), Part I below, and complete the preparatory exercises *before* the start of the lab. Provide answers to the preparatory exercises on the same answer sheet as used for reporting the computer exercise. General rules for the preparatory exercises and the computer exercises:

- Each student should hand in individually completed solutions. (Note that the exam will likely contain questions on the preparatory exercises and the lab material!)
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or code to other students.

Introduction

In this theme we study a rocket launch with the aim to find the optimal fuel burning strategy for rocket thrusters. That is, given a certain rocket configuration and a fixed amount of total available fuel, we wish to find the fuel burning rate as as a function of time from takeoff in order for the rocket to reach an altitude as high as possible. The fuel burning strategy plays a critical role in the success of a rocket launch. Since we are on a low budget, we choose to use a single-stage rocket system—these systems are much cheaper to build and maintain than multi-stage systems. Moreover, it is easier to model a single-stage system since, in contrast to the popular multi-stage rockets used in space flight, our rocket cannot drop parts corresponding to used engines during misson.

A straightforward but expensive way to solve this problem is to build rockets and make experiments. However, the low budget will necessitate a reliance of computer simulations instead. In this theme, we will use numerical methods to simulate the rocket launch given a certain fuel burning strategy while manually modifying the strategy. There are various techniques that can be used to let the computer also search for the optimal strategy in a systematic way. However such methods are out of scope for the present course.

Problem Specification

Before even trying to find a fuel burning strategy, we need to be able to compute the maximum altitude that can be reached given a certain fuel burning strategy. Let m and v denote the mass and speed of the rocket, respectively, and μ the fuel consumption in kg/s; it is μ as a function of time that we will manipulate in order to maximize the final altitude. We assume that the dynamics of the rocket is given by the simplified model

$$m'v + mv' = c_1\mu - c_2 v|v| - mg. \qquad (1)$$

Equation (1) is simply Newton's second law: the time derivative of the momentum (the left side) equals the total forces on the rocket (the right side). Note that the left-side momentum derivative is not just "mass times acceleration" as for simple point mass systems; here we also obtain the term $m'v$ since the mass of the rocket changes with time due to the burning of fuel.

There are three contributions to the forces on the rocket, as modelled by the right side of equation (1). The first term is the engine thrust (Swe. *motorkraft*), which we assume to be proportional, with constant $c_1 = 1000$ m/s, to the fuel consumption μ. The second term is the force due to air resistance. This force is assumed to be proportional, with constant $c_2 = 1/3$ kg/m, to the square of the velocity, a scaling law that is typical for aerodynamic forces ("velocity-squared law"). Note also that the air resistance force, as written above, will always be directed opposite to the velocity! The third and final force term represents the gravitational force, where $g = 9.82$ m/s^2 is the gravitational acceleration.

The rocket can burn at most 10 kg fuel per second, thus $\mu \in [0, 10]$ kg/s, and the mass of the rocket m decreases with the mass of the burnt fuel, that is,

$$m' = -\mu.$$

Without fuel the rocket weights 100 kg. Initially, the fuel weights 900 kg which yields a total weight of 1000 kg at the initial time, that is, $m(0) = 1000$ kg. Since we are interested in shooting the rocket as high as possible we also wish to monitor the height h above sea level. Assuming that we are firing the rocket from rest in Umeå, $h(0) = 14$ m and $v(0) = 0$ m/s. Also recall that it holds that

$$h' = v.$$

Exercise

1. Set up an ODE for the rocket problem in standard form using $\mathbf{u} = [h, v, m]^\mathsf{T}$.