Institutionen för datavetenskap
Umeå universitet

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Numeriska metoder för civilingenjörer

Theme: Network models and linear systems
Part I

Study the theory (the lecture notes and relevant sections in the book), Part I below, and complete the preparatory exercises *before* the start of the lab. Provide answers to the preparatory exercises on the same answer sheet as used for reporting the computer exercise. General rules for the preparatory exercises and the computer exercises:

- Each student should hand in individually completed solutions. (Note that the exam will likely contain questions on the preparatory exercises and the lab material!)
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or code to other students.

Theme introduction

Various kinds of *network problems* arise again and again in many applications and yields linear systems of equations with particular properties. Figure 1 shows an example of a small network. Networks are modeled mathematically using *graph theory*.[1] A *graph* consists of a collection of *vertices* (or *nodes*; *noder* in Swedish) and *edges* (*kanter* or *bågar* in Swedish) that pairwise connect the vertices. The network in figure 1 is a *directed graph*[2] that could represent an electrical network connecting components like resistors, capacitors, inductors, and electric sources. The network could alternatively illustrate transportation routes for goods between cites. More examples of systems that can be modeled using graphs are truss networks, computer networks, and the World Wide Web.

Here, we will take a closer look at networks of *water pipes*. Assume that each of the five numbered edges in figure 1 represents a pipe through which water may flow. The vertices represent sources (the water company) or sinks (residential houses, say) of water. Assuming that we know the amount of water per time unit that is supplied or removed at each vertex, we like to determine the water pressures that are needed at the various vertices. This is a relevant question when dimensioning pumps and pipes. We will describe a general methodology to solve this problem under simplified assumptions.

The arrows in figure 1 indicate sign conventions for positive flux through the pipes. For example, water flows from vertex 1 to vertex 3 at a rate of u_1 m^3/s ($u_1 < 0$ indicates net flow from vertex 3 to vertex 1). Moreover, we know the rates s_i (in m^3/s) at which water is supplied or removed at vertex i; $s_i > 0$ means that water is added and $s_i < 0$ that water is removed from the network.

We assume that there are no leaks in the network. Thus, the flux of water *out* of vertex i, $-s_i$ must be exactly equal the sum of the fluxes that are coming *into* vertex i through the edges

[1] See for instance Wikipedia's article `http://en.wikipedia.org/wiki/Graph_(mathematics)`
[2] `http://en.wikipedia.org/wiki/Directed_graph`

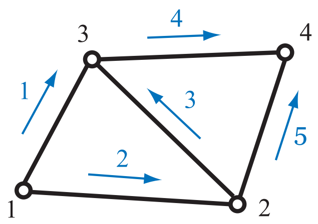

Figure 1: An example of a network in terms of a directed graph with 4 vertices and 5 edges

that are attached to vertex i. The flow of water in the network in figure 1 will then satisfy the equations

$$\begin{aligned} -u_1 - u_2 &= -s_1 && \text{(at node 1)} \\ u_2 - u_3 - u_5 &= -s_2 && \text{(at node 2)} \\ u_1 + u_3 - u_4 &= -s_3 && \text{(at node 3)} \\ u_4 + u_5 &= -s_4 && \text{(at node 4).} \end{aligned} \tag{1}$$

Note that the plus or minus signs of equation (1) depend on the sign conventions; we have (arbitrary) assigned directions as in figure 1 and defined positive sources as adding to the network.

Since there are no leaks in the system, and since water is hard to compress, it holds that everything that goes in must come out:

$$s_1 + s_2 + s_3 + s_4 = 0. \tag{2}$$

Equation (1) can be written in the matrix form

$$Bu = -s, \tag{3}$$

where

$$B = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}. \tag{4}$$

Matrix B is called the *incidence matrix*[3] of the directed graph and give a complete description of how the network is connected—its “topology”. The incidence matrix reveals nothing, however, about the coordinates of the vertices or the edges, an information that is not needed for the present problem. An incidence matrix for a directed graph has as many rows as vertices in the graph and as many columns as edges in the graph. Every column represents thus an edge and has $+1$ and -1 at the rows that correspond to the vertices in which there are inflow and outflow, respectively. All other entries in B are zero. Note that the incidence matrix typically has more columns that rows, since a graph typically have more edges than vertices.

Denote the pressure (in Pa) at vertex i by p_i. The flux of water through a particular edge depends on the *difference* of pressure between the two vertices that the edge connects. We assume a simple linear relationship so that the flux is proportional to the pressure difference:

$$\begin{aligned} u_1 &= \epsilon_1(p_1 - p_3) && \text{(edge 1),} \\ u_2 &= \epsilon_2(p_1 - p_2) && \text{(edge 2),} \\ u_3 &= \epsilon_3(p_2 - p_3) && \text{(edge 3),} \\ u_4 &= \epsilon_4(p_3 - p_4) && \text{(edge 4),} \\ u_5 &= \epsilon_5(p_2 - p_4) && \text{(edge 5),} \end{aligned} \tag{5}$$

[3] `http://en.wikipedia.org/wiki/Incidence_matrix`

where the coefficients $\epsilon_i \geq 0$ denotes the "conductivity" of that pipe; a property that can be determined by experiments, for instance. When writing equation (5) (which corresponds to Ohm's law for electrical circuits) in matrix form,

$$u = -EB^T p, \tag{6}$$

where

$$E = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_4 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_5 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \tag{7}$$

we see that the transpose of the incidence matrix also appears in the pressure–flux relation.

To further simplify the modeling, we consider a case where the pressure at vertex 4 is held at the constant valus $p_4 = 0$. The unknown pressures p_1, p_2, and p_3 will then denote the pressure relative to the pressure at vertex 4, analogous to the concept of zero-potential "ground" in electrical circuits. Also, we let s_1, s_2, and s_3 be given arbitrarily and define $s_4 = -s_1 - s_2 - s_3$ so that equation (2) is automatically satisfied. These simplifications imply that we may delete the fourth row in matrix B (since we already know what happens at vertex 4!) and remove s_4 and p_4 from the problem (since these are known). That yields one less equation (in the system $Bu = -s$) and one less unknown (in $u = -EB^T p$), and we obtain

$$\bar{B}u = -\bar{s} \tag{8a}$$

$$u = -E\bar{B}^T \bar{p}, \tag{8b}$$

where

$$\bar{B} = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix}, \quad \bar{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3. \end{pmatrix}. \tag{9}$$

Multiplying both sides of equation (8b) from the left with $-\bar{B}$ and adding the resulting equation to equation (8a) yields the following equation in the reduced unknowns $\bar{p}$:

$$\bar{B}E\bar{B}^T \bar{p} = \bar{s}. \tag{10}$$

Preparatory exercises for lab

1. Compute by hand the LU-factors of matrix

$$A = \begin{pmatrix} 1 & 1.5 & 1 & -1 \\ 2 & 1 & 3 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{pmatrix}.$$

2. Consider equation system (10) in the particular case of the network of figure 1.

   (a) Explicitly compute the left-hand-side matrix for arbitrary values of the conductivity parameters $\epsilon_1, \ldots, \epsilon_5$.

   (b) Compute by hand the LU factorization of the above matrix for the particular conductivities $\epsilon_1 = 1, \epsilon_2 = 2, \epsilon_3 = 2, \epsilon_4 = 3, \epsilon_5 = 4$.

   (c) Use the LU-factorization you computed above to solve the system by hand for the right-hand side vector $s = (0, 4, 3)^T$.