Institutionen för datavetenskap
Umeå universitet

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Numeriska metoder för civilingenjörer

Theme: Roundoff and population modeling
Part I

Study the theory (the lecture notes and relevant sections in the book), Part I below, and complete the preparatory exercises *before* the start of the lab. Provide answers to the preparatory exercises on the same answer sheet as used for reporting the computer exercise. General rules for the preparatory exercises and the computer exercises:

- Each student should hand in individually completed solutions. (Note that the exam will likely contain questions on the preparatory exercises and the lab material!)
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or code to other students.

The logistic map

Suppose we want to model the population of an animal species. Each year the population increases by a factor R, the *growth rate*. In other words, if the number of animals is N_n in year n, then it will be RN_n in year $n+1$, that is,

$$N_{n+1} = RN_n. \tag{1}$$

To make the model more realistic, we allow the growth rate to be a function of the population size, that is, $R = R(N_n)$. We assume that there is a maximum number of animals that sustainably can be maintained by the environment. This number, the *carrying capacity* of the environment, is denoted by $N_{\max}$. Moreover, we expect the growth rate to approach zero as the population reaches the carrying capacity, that is, $R(N_n) \to 0$ as $N_n \to N_{\max}$. A simple function that satisfies this criterion is

$$R(N_n) = r\left(1 - \frac{N_n}{N_{\max}}\right),$$

where r is a constant. Substituting this formula into (1), we obtain

$$N_{n+1} = r\left(1 - \frac{N_n}{N_{\max}}\right)N_n. \tag{2}$$

To simplify (2), we divide by the carrying capacity $N_{\max}$ and set $x_n = N_n/N_{\max}$, so that

$$x_{n+1} = rx_n(1 - x_n) \tag{3}$$

with $x_n \in [0,1]$. The graph of $f(x) = rx(1-x)$ is a parabola with a maximum value of $r/4$ at $x = 1/2$. We restrict the parameter r to the range $[0,4]$ so that f maps the interval $[0,1]$ into itself; this function, depicted in Figure 1, is called the *logistic map*.

Sequences $x_0, x_1, \ldots,$ generated by recursion formulas such as (3) are called *orbits*. The character of the orbits generated by the logistic map depends strongly on the value of r. Recursion formula (3) is a particularly simple example of a system that can show a *chaotic* behavior; for particular values of r, the orbits will be aperiodic (or irregular) and highly sensitive to changes in the initial conditions, that is, slight variations in the initial population yields dramatically different evolution of the species.[1]

[1] Steven H. Strogatz, Nonlinear dynamics and chaos, Perseus Books, 1994.

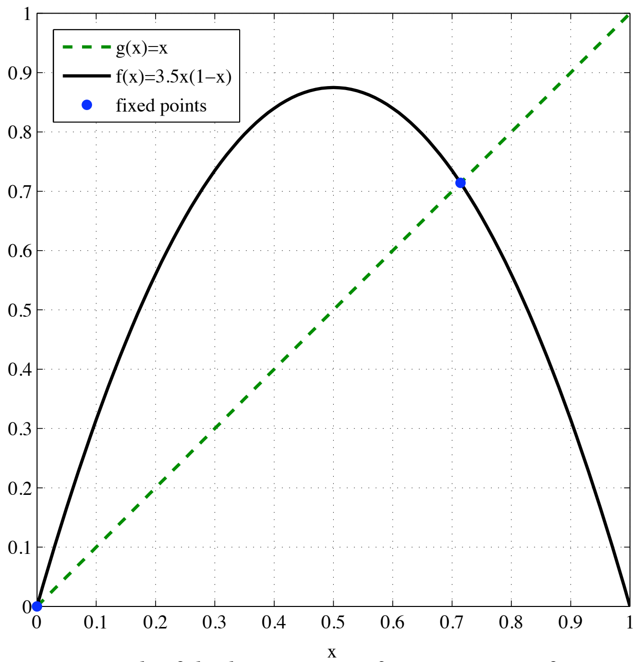

Figure 1: Graph of the logistic map $f(x) = rx(1-x)$ for $r = 3.5$

Preparatory exercises

1. A point x_* is called a *fixed point* of a function f if $f(x_*) = x_*$. The fixed points of the logistic map for $r = 3.5$ are marked with dots in Figure 1. Find all fixed points of the logistic map for arbitrary r. What is the significance of the fixed points of f for the orbits generated by recursion formula (3)?

2. Explain why the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

has a finite sum in floating point arithmetic.