Institutionen för datavetenskap
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Numeriska metoder för civilingenjörer

Review exercises for Themes 4 and 5

1 Theme 4

1. True or false: if a solution y to an initial value problem for an ODE satisfies $y(t) \to +\infty$ as $t \to +\infty$, then the ODE is necessarily unstable.

2. Classify the following ODEs as stable, asymptotically stable, or unstable with respect to initial values.

   (a) $y' - y = -t^2$, (b) $y' - y = t^2$, (c) $y' + y = t^2$; (d) $y' = t^2$.

3. Write following equations in the standard form for initial value problems.

   (a) Duffing's equation:
   $$y'' + \delta y' + \sigma(y^3 - y) = \gamma \sin \omega t$$

   (b) Blasius' equation:
   $$f''' + \frac{1}{2} f f'' = 0$$

   (c) Van der Pol's equation:
   $$y'' - y'(1 - y^2) - y = 0.$$

4. Shortly explain what is meant by the following terminology in connection with the numerical solution of initial-value problems for ordinary differential equations:

   (a) *implicit* and *explicit* methods;
   (b) *truncation error*;
   (c) *order of accuracy*.

5. True or false: can a numerical method be unstable when it is applied to an ODE which is stable with respect to initial conditions?

6. Implicit numerical methods for the solution of initial-value problems for ordinary differential equations typically have a much larger range of time steps for which the method is stable compared to explicit methods. Why then are implicit methods not always used?

7. The initial value problem for the ordinary differential equation $y' = f(t, y)$ can be solved numerically using the so-called α scheme:
   $$y_{k+1} = y_k + \Delta t\big[\alpha f(t_{k+1}, y_{k+1}) + (1 - \alpha) f(t_k, y_k)\big],$$
   where $0 \le \alpha \le 1$.

   (a) What methods are obtained when selecting $\alpha = 0$, 1, and 1/2, respectively?
   (b) Determine the order of accuracy for the scheme for all $0 \le \alpha \le 1$.
   (c) For all $0 \le \alpha \le 1$, determine the stability condition on the negative real axis, that is, the condition of stability for the scheme when applied to the equation $y' = \lambda y$ for $\lambda < 0$.
   (d) What changes have to be done to the α-scheme in order to obtain Heun's method?

8. The initial value problem for the ordinary differential equation $y' = f(t, y)$ can be numerically solved with the scheme

$$y_{k+1} = y_k + \frac{\Delta t}{2}\left[3f(t_k, y_k) - f(t_{k-1}, y_{k-1})\right].$$

   (a) Is the method explicit or implicit?
   (b) Derive the order of accuracy for the method.

9. What is meant by a *stiff* system of ODEs? Why are implicit methods often recommended for stiff system of ODEs?

2 Theme 5

1. Let $(x_0, y_0), \ldots, (x_n, y_n)$ be a set of pair of numbers.

   (a) Give a condition under which the point sets can be interpolated by polynomials.
   (b) What problem can occur when interpolating arbitrary point sets with polynomials?

2. You have been given the task of providing a drawing of a church vault (*valv*) by interpolating a set of 25 measured x- and y-coordinates. Suggest a suitable computational technique to interpolate the points in order to plot the shape of the vault.

3. A function f is defined on the interval $[0, 1]$, but the function values are known only at the points in the table. The table suggests that the underlying function has a maximum within the interval. It is likely that the maximum is not exactly at one of the given points.

   (a) When interpolating a suitable subset of the points with a polynomial in order to estimate the exact location of the maximum, which points and which order of the polynomial is appropriate to use?

   (b) Carry out the interpolation and estimate the location of the maximum.

x	$f(x)$
0.0	0.1206
0.1	0.2506
0.2	0.4329
0.3	0.6532
0.4	0.8748
0.5	1.0441
0.6	1.1136
0.7	1.0656
0.8	0.9213
0.9	0.7295
1.0	0.5427

4. Derive a formula for numerical computation of the double integral

$$\int_0^1 \int_0^1 f(x, y)\,\mathrm{d}x\mathrm{d}y$$

   through repeated use of the trapezoidal method with equidistant intervals first in one coordinate direction and then in the other.

5. A flight is planned between Stockholm and New York (distance 7800 km) using an airplane with a nominal cruising speed of 700 km/h. The effective speed (the speed relative to ground) varies depending on the wind direction. Using meteorological data, the effective speed has been estimated according to the table.

   The flight time for a flight from x_1 to x_2 is given by the integral

$$T = \int_{x_1}^{x_2} \frac{1}{v(x)}\,\mathrm{d}x$$

x (km)	v (km/h)
0	750
1300	680
2600	630
3900	640
5200	690
6500	760
7800	830

   Estimate the flight time by numerically computing the above integral.