Review exercises for quiz

1 Theme 1

1. What is the definition of machine epsilon ϵ_M?

2. What is the largest error that can occur when rounding a real number x to a floating point number $fl(x)$ (assuming that over- or underflow does not occur)? Estimate the error in the floating point representation of the number π when $\epsilon_M = 2^{-52}$ (IEEE 754 double precision).

3. As part of an algorithm, there is a need to test whether $f(x) = a$, where $f(x)$ and a are real numbers. The values of $f(x)$ and a are stored as floating point numbers `f` and `a` in a Matlab implementation of the algorithm. How should the test be implemented?

4. Which error, the rounding error or the discretization error, usually dominates in scientific computations?

5. Name two cases in which rounding errors may have a large impact on the result of a computation that uses floating point numbers.

6. What is meant by underflow in floating point calculations? Give example of a calculation that yields `NaN` in the IEEE 754 standard.

7. Give an example of a calculation that may cause cancellation of significant digits.

8. Assume that we wish to approximate the derivative with the so-called central difference

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

   What kind of errors dominate for different sizes of h?

9. Why does the divergent series $\sum_{k=1}^{\infty} 1/k$ has a finite sum in floating point arithmetic?

2 Theme 2

1. *Without* the use of matrix inverse (`inv`), write a Matlab expression that computes the vector

$$x = B^{-1}(2A + I)(C^{-1} + A)b,$$

   where A, B, and C are given $n \times n$ matrices, with B and C nonsingular, and b is a given n vector.

2. Assume that the LU factorization of a matrix A of size 5000-by-5000 takes 11 s. Estimate how long time the backward and forward substitutions will take. (You may ignore delays due to startup and memory operations.)

3. When solving linear systems of equations $Ax = b$ in Matlab, the common recommendation is to write `x=A\b` and not `x=inv(A)*b`. Why?

4. What is the recommended strategy when solving several linear system with the same matrix but multiple right-hand sides?

5. Assume that the LU factorization of a matrix is known, that is, upper and lower triangular matrices L and U that satisfy $A = LU$ are known. How can this information be used in order to solve the linear system $A^T x = b$?

6. Does the conditioning of a linear system depend on the computer algorithm that is used to solve the problem?

7. Compute matrix norms $\|A\|_\infty$, $\|A\|_1$ for

$$A = \begin{pmatrix} -1.1 & 2 & 0.5 & 1 \\ 2 & 3 & -3 & 0.5 \\ 1.5 & 0.9 & 0 & 0.1 \\ 0.5 & 0.6 & -0.6 & 0.7 \end{pmatrix}.$$

   Explain how the matrix norm $\|A\|_2$ could be computed.

8. Classify following matrices as *well conditioned* (a low condition number) or *ill conditioned* (a high condition number):

$$\begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{pmatrix}, \quad \begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{10} \end{pmatrix}, \quad \begin{pmatrix} 10^{-10} & 0 \\ 0 & 10^{-10} \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

9. (a) Show that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   cannot be LU-factorized; that is, there are no matrices L and U (with the structure we have discussed in the course) such that $A = LU$.

   (b) What action does software for solution of linear system (such as Matlab) take in order to be able to factor matrices like A?

10. (a) Perform LU factorization, without pivoting, of the matrix

$$A = \begin{pmatrix} 1 & 0.5 & 1.5 & -1 \\ 2 & 3 & 2 & -2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 2 \end{pmatrix}.$$

   (b) What problems can occur in a computer implementation of LU factorization if row pivoting is not performed?

11. (a) LU factorize the matrix

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$$

   without row pivoting. Use the L and U factors to solve the equation system $Ax = b$ with $b = (1, 1, 1)^T$.

   (b) Now assume that vector b above is not exactly known but rounded to 3 correct decimals (that is, the magnitude of the rounding error in each component of b is not larger than 5×10^{-4}). Moreover, it holds that $\|A^{-1}\|_\infty = 7$. How large will then the error in the computed solution be in the worst case?

3 Theme 3

1. What is meant by *linear convergence rate* for a method to solve nonlinear systems of equations?

2. Assume that the errors in an iterative method develop as follows:

   (a) $10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}$

   (b) $10^{-3}, 10^{-5}, 10^{-7}, 10^{-9}$

   What are corresponding convergence rates?

3. How many iterations does Newton's method require to solve a *linear* system of equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$? Motivate.

4. List one advantage of fixed-point iterations compared to Newton's method for solving nonlinear equations. List one advantage with Newton's method.

5. Write up Newton's method for the solution of the minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}),$$

   where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function of n variables. *Note:* Optional problem that requires additional concepts from multivariate calculus!