Institutionen för datavetenskap
Umeå universitet

December 8, 2009
Numeriska metoder för civilingenjörer

Theme: Quadrature and Interpolation
Part I

Study the theory (the lecture notes and relevant sections in the book), Part I below, and complete the preparatory exercises *before* the start of the lab. The preparatory exercises need to be completed to get a passing grade on the lab, so show them to the teacher in order to verify that they are completed. At the beginning of the lab, Part II of the theme, the computer exercises, will be handed out and uploaded to the home page of the course. General rules for the preparatory exercises and the computer exercises:

- Each student should hand in individually completed solutions.
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or code to other students.

Introduction

The problem this time is to compute the center of mass of a homogeneous two dimensional object with density ρ (kg/m^2), occupying a region $\Omega \subset \mathbb{R}^2$. This can be done in two steps. First, we can compute the mass of the object

$$m = \int_\Omega \rho \, \mathrm{d}A,$$

and then the center of mass of the object,

$$\bar{x} = \frac{1}{m}\left(\int_\Omega x_1 \rho \, \mathrm{d}A, \int_\Omega x_2 \rho \, \mathrm{d}A\right).$$

Note that since the object is homogeneous, the density ρ is constant and will be canceled in the division when computing the center of mass. Thus, for simplicity, we set $\rho \equiv 1$.

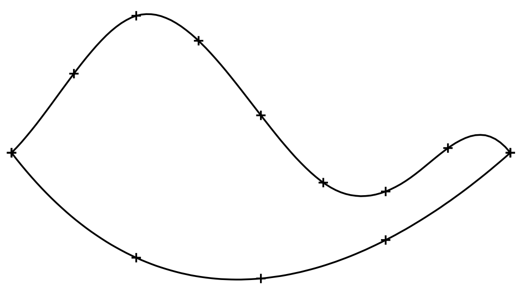

Figure 1: Plan sketch of object for which we are interested in finding the center of mass.

What is complicating our work this time is that we do not know the exact shape of the object. We have a hand-drawn plan sketch (Figure 1), where some points are specified. Fortunately, the shape of the object can be specified using two functions in the following manner

$$\Omega = \left\{ x \in \mathbb{R}^2 \mid a \le x_1 \le b,\, f_1(x_1) \le x_2 \le f_2(x_1) \right\},$$

where the graphs of f_1 and f_2 correspond to the lower and upper boundaries of the object, respectively.

Exercises

1. Use the above assumptions to reduce the integrals required to compute the center of mass to one dimensional integrals.

2. Consider the general quadrature formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \alpha_1 f(-\beta) + \alpha_2 f(\beta).$$

   Derive conditions on the constants α_1, α_2, and β so that the formula is exact for all

   (a) constant functions $f(x)$,

   (b) first order polynomials $f(x)$,

   (c) second order polynomials $f(x)$.

   What is the highest-order polynomial that can be integrated exactly when using the constants derived in (c)?

3. We want to study a function that is sampled or measured only at a finite number of points. One way to approximate the derivative of the function is to interpolate the discrete data points using a polynomial and then differentiate this polynomial. Is this a good method? Motivate!