Institutionen för datavetenskap
Umeå universitet

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Numeriska metoder för civilingenjörer

# Theme: Rocket launches and ODEs
# Part II

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Rules:

- During the lab, verify with the teacher that the preparatory exercises in Part I are completed.
- Each student should hand in individually completed solution.
- Use the provided answer sheet and add printouts of pictures and source code as requested.
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or your code to other students.
- A correct solution submitted at the latest **December 8** is worth 2 bonus point on the final exam. An almost correct solution submitted in time will also entitle to bonus points, provided the solution is corrected before the final exam.

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1. Implement your own version of Heun's method to solve the system of ODEs

$$\begin{aligned} \mathbf{y}' &= \mathbf{f}(t,\mathbf{y}) \qquad t > 0, \\ \mathbf{y}(0) &= \mathbf{y}_0. \end{aligned} \tag{1}$$

Make the implementation in a separate MATLAB function, `heun.m`. The function header should look like as below,[1] and the function should follow the described behavior.

```
function [t,y] = heun(f,tspan,y0,h)
%HEUN solve differential equation with Heun's method
%   [T, Y] = HEUN(F,TSPAN,Y0,H) with TSPAN = [T0, TFINAL] integrates
%   the system of differential equations y' = f(t,y) from time T0 to
%   TFINAL with initial conditions Y0 employing a constant time step
%   H. F is a function handle. For a scalar T and a vector Y, F(T,Y)
%   must return a column vector corresponding to f(t,y). Each row in
%   the solution array Y corresponds to a time in the vector T.
```

The above function follows the same standard as Matlab's build in ODE solvers. Note the dimensions of the matrices! Assume that system (1) is of dimension $m$ (that is, the

[1] The comment rows just after the function declaration are displayed when you type the command `help heun`. The header can be downloaded from the course homepage.

length of vectors $\mathbf{y}$ and $\mathbf{f}$ is $m$) and that `heun` has been solved for $N$ time steps. The output matrix $\mathbf{Y}$ should then have the shape

$$\mathbf{Y} = \begin{pmatrix} -\mathbf{y}_0- \\ -\mathbf{y}_1- \\ \vdots \\ -\mathbf{y}_N- \end{pmatrix},$$

where each row is an $m$-vector. The first row contains the initial condition, and each of the following rows the numerical solution at times $t_1$, $t_2$, .... However, the Matlab function `f` that implements the right-hand side $\mathbf{f}(t, \mathbf{u})$ must return a *column* vector! It may seem unpractical to output the solution vectors row wise when the right-hand-side vector must be a column vector, but this is the standard chosen by Matlab. **Warning:** the prime operator `'` used as `A'` does not only transpose `A`, it also complex conjugates all elements. To only transpose `A`, without complex conjugation, use `A.'` or `transpose(A)`.

2. Test your implementation of Heun's method on the equation $y' = \alpha y$, where $\alpha = -0.1 + 5\mathrm{i}$, from time $T_0 = 0$ to time $T = 30$ with initial condition $y(0) = 1$. Plot your numerical solution together with the exact solution.

3. Again, integrate the equation $y' = \alpha y$, where $\alpha = -0.1 + 5\mathrm{i}$, from time $T_0 = 0$ to time $T = 30$ with initial condition $y(0) = 1$ using your implementation of Heun's method. Try using different time steps $h$ to

   (a) experimentally find the stability limit,

   (b) experimentally estimate the order of accuracy by computing the absolute value of the difference between your numerical solution and the exact solution at time $T = 30$.

4. Use your methods as well as MATLAB's build in function `ode45` to find the maximum height the rocket reaches with the strategy to burn off fuel as fast as possible, that is

$$\mu = \begin{cases} 10 \text{ kg/s} & \text{if } t \leq 90 \text{ s}, \\ 0 \text{ kg/s} & \text{if } t > 90 \text{ s}. \end{cases}$$

5. Try to find a better fuel burning strategy. Recall that there is only 900 kg of fuel at the launch! Describe your strategy, how high does your rocket fly?