Theme: Rocket launches and ODEs
Part I

Study the theory (the lecture notes and relevant sections in the book), Part I below, and complete the preparatory exercises *before* the start of the lab. The preparatory exercises need to be completed to get a passing grade on the lab, so show them to the teacher in order to verify that they are completed. At the beginning of the lab, Part II of the theme, the computer exercises, will be handed out and uploaded to the home page of the course. General rules for the preparatory exercises and the computer exercises:

- Each student should hand in individually completed solutions.
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or code to other students.

Introduction

In this theme we study a rocket launch with aim to find the optimal fuel burning strategy for rocket thrusters. That is, given a certain rocket configuration we wish to control the amount of fuel being burnt (as a function of time from takeoff) so that the rocket reaches an as high altitude as possible. The fuel burning strategy plays a critical role in the success of a rocket launch. Since we are on a low budget, we choose to use a single stage rocket system—these systems are much cheaper to build and maintain than multi stage systems. Moreover, it is easier to model a single stage system; since, in contrast to the popular multi stage rockets used in space flight, our rocket cannot drop parts corresponding to used engines during the flight.

A naive way to solve this problem is to build rockets and make experiments. This is indeed a very expensive and inefficient strategy. As a cost-and-time efficient complement, we instead choose to create a computer model of the experiments. In this theme, we will use numerical methods to simulate the rocket launch given a certain fuel burning strategy while manually modifying the strategy. There are various techniques that can be used to let the computer also search for the optimal strategy in a systematic way. However such methods are out of scope for the present course.

Problem Specification

Before even trying to find a fuel burning strategy, we need to be able to compute the maximum altitude that can be reached given a certain fuel burning strategy. Let m and v denote the mass and speed of the rocket, respectively, and μ the controllable fuel consumption in kg/s. We assume that the dynamics of the rocket is given by the simplified model

$$m'v + mv' = c_1\mu - c_2 v|v| - mg, \tag{1}$$

where the right hand side denotes the acceleration forces from the engine and reaction forces from air resistance and gravity. The first force term models the acceleration forces from the engine, for our rocket the constant $c_1 = 1000$ m/s is used. The second force term is a velocity squared law for the air resistance, for our rocket the constant $c_2 = \frac{1}{3}$ kg/m. The final force term represents the gravitational forces and $g = 9.82$ m/s^2 represents the gravitational acceleration. Our rocket can burn at most 10 kg fuel per second, thus $\mu \in [0, 10]$ kg/s and the mass of the rocket m decreases with the mass of the burnt fuel, that is,

$$m' = -\mu.$$

Without fuel the rocket weights 100 kg. Initially, the fuel weights 900 kg which yields a total weight of 1000 kg at the initial time, that is, $m(0) = 1000$ kg. Since we are interested in shooting the rocket as high as possible we also wish to monitor the height h above sea level. Assuming that we are firing the rocket from rest in Umeå, $h(0) = 14$ m and $v(0) = 0$ m/s, and for completeness we recall that

$$h' = v.$$

Exercise

1. Set up an ODE for the rocket problem in standard form using $\mathbf{u} = [h, v, m]^{\mathsf{T}}$.