Institutionen för datavetenskap
Umeå universitet

6 november 2009
Numeriska metoder för civilingenjörer

Theme: Roundoff and population modeling
Part II

Rules:

- During the lab, verify with the teacher that the preparatory exercises in Part I are completed.
- Each student should hand in individually completed solution.
- Use the provided answer sheet and add printouts of pictures and source code as requested.
- You may discuss the problem among fellow students. If you receive considerable help from someone, say so in your solutions.
- Do not copy solutions or code from others. Do not lend your solution or your code to other students.
- A correct solution submitted at the latest **November 11** is worth 2 bonus point on the final exam. An almost correct solution submitted in time will also entitle to bonus points, provided the solution is corrected before the final exam.

Computer exercises

1. (a) In MATLAB, the command `eps` returns machine epsilon ϵ_M. More precisely, `eps(x)` returns the distance from `abs(x)` to the next larger in magnitude floating point number (of the same precision as `x`), and `eps` is a shorthand notation for `eps(1)`. Try the command. Of what order of magnitude is ϵ_M?

   (b) Is it possible to represent numbers smaller than ϵ_M? If so, which is the smallest (positive greater than zero) number that can be represented using IEEE 754 double precision?

   (c) What does `NaN` and `Inf` mean? Try some operations with `NaN`, `Inf`, and ordinary numbers such as `Inf-Inf`, `Inf+Inf`, `1/Inf`, `-1/0`, `NaN<-Inf`, `NaN>-Inf`, and `2*Inf==Inf` .

2. Predict the final value of `x` in each of the following statements. Motivate!

   (a)
   ```
   x=1;
   while 1+x>1
     x = x/2;
     disp(x);
   end
   ```
   (b)
   ```
   x=1;
   while x+x>x
     x = x/2;
     disp(x);
   end
   ```
   (c)
   ```
   x=1;
   while x+x>x
     x = 2*x;
     disp(x);
   end
   ```
   Test if your predictions are correct.

3. Use MATLAB to generate the sequence $x_1, x_2, \ldots, x_{60}$, defined by

$$x_n = \frac{7}{3}x_{n-1} - \frac{2}{3}x_{n-2}$$

with initial conditions $x_1 = 3$ and $x_2 = 1$. Plot the values using `semilogy`. The exact solution to the recursion with the above starting values is $x_n = 3^{2-n}$ and the general solution to the recursion is $x_n = c_1 2^n + c_2 3^{-n}$. Comment your results!

4. Sometimes, computations can be rearranged so that they are less prone to rounding errors. One way of approximating π (Archimedes) is by calculating perimeter of polygons circumscribing a circle with diameter 1. Starting with hexagons and doubling the sides at each step the length of each side of the polygon follow the recursion $t_0 = 1/\sqrt{3}$

$$t_{n+1} = \frac{t_n}{\sqrt{t_n^2+1}+1}, \text{ or alternatively } t_{n+1} = \frac{\sqrt{t_n^2+1}-1}{t_n}$$

The approximation of π in step n is given by $2^n \cdot 6t_n$. Try both versions and explain why one of the above recursions is less error prone.

5. The derivative f' of a smooth function f at a point x is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This limit can be approximated by the finite difference approximation

$$D^+_{\Delta x} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{1}$$

where $\Delta x > 0$ is a real number. In exact arithmetic it holds that $D^+_{\Delta x} f(x) \to f'(x)$ as $\Delta x \to 0$. However, we are working with floating point arithmetic so due to cancelations one has to be careful when choosing the step Δx. Let $f(x) = e^x$ and approximate f' using the finite difference approximation (1) at $x = 1$ using $\Delta x = 10^{-n}$ for $n = 1, 2, \ldots$ and plot the error $|D^+_{\Delta x} f(x) - f'(x)|$ as a function of Δx.

   If you have time, try using other functions. Can you find a rule of thumb on how to choose Δx to obtain a good approximation of the derivative?

6. Write a MATLAB function `logmap(x0,r,k)` that returns the vector $\boldsymbol{x} = [x_0, x_1, x_2, ..., x_k]^T$ for the logistic map, where `x0` is the initial condition, `r` is the constant r, and `k` is the number of iterations. Plot the results as time series of x_n against n for some sensible initial value $x_0(= 0.1)$, $k = 100$ iterations, and $r \in \{2.8, 3.3, 3.5, 3.9\}$. Characterize the orbits $\boldsymbol{x}$ as regular or irregular.

7. Write a MATLAB function that returns $\boldsymbol{\delta} = [\delta_1, \delta_2, ..., \delta_k]^T$ where $\delta_i = |x_i - y_i|$ with x_i the values returned by iterating the logistic map and y_i the values returned by iterating $y_{n+1} = ry_n - ry_n^2$. Plot $\boldsymbol{\delta}$ for some sensible initial values $x_0 = y_0(= 0.1)$, $k = 200$ iterations, and $r \in \{3.3, 3.9\}$. Give an interpretation of the results.

8. In spite of the above, roundoff errors rarely cause problems in practical calculations. There are, however, two instances as where roundoff effects may cause problems:

   - Problems that are inherently sensitive to disturbances in data regardless of which numerical algorithm that is used (ill-conditioned problems).
   - Algorithms sensitive to round-off even when applied to a well-conditioned problem (numerically unstable algorithms). Such algorithms should be avoided if possible!

   Classify problems 4, 5, and 7 above as ill-conditioned or numerically unstable.