

Laboration 2: Trajectory of a cannon ball

This computer laboration is an example of how numerical methods can be applied to solve problems in physics. The task is to apply Euler's (explicit) method to calculate the trajectory of a cannon ball, including the effects of air resistance, using MATLAB.

1 Summary of Euler's method

- The first step is to divide the time interval that is of interest, e.g. $t \in [t_a, t_b]$, into a finite number $(N + 1)$ of points $[t_0, t_1, t_2, \dots, t_N]$, where $t_0 = t_a$ and $t_N = t_b$. The distance between the points are $\Delta t = (t_b - t_a)/N$.
- An ordinary differential equation of the type

$$\frac{dx(t)}{dt} = f(t) \quad (1)$$

$$x(t_a) = X \quad (2)$$

where $f(t)$ is a given function, can be solved numerically with Euler's (explicit) method. The algorithm can be summarized as

$$x_{n+1} = x_n + f_n \Delta t \quad (3)$$

where x_n and f_n are the value of $x(t)$ and $f(t)$ at time $t = t_a + n\Delta t$.

- A second order ordinary differential equation, for example

$$\frac{d^2x(t)}{dt^2} = f(t) \quad (4)$$

$$x(t_a) = X, \quad \left. \frac{dx(t)}{dt} \right|_{t=t_a} = V \quad (5)$$

can be solved by introducing an auxiliary function $v(t) = \frac{dx(t)}{dt}$. The problem then becomes to solve *two coupled* first order equations, one for $x(t)$ and one for $v(t)$. The algorithm, in this case, can be summarize as

$$x_{n+1} = x_n + v_n \Delta t \quad (6)$$

$$v_{n+1} = v_n + f_n \Delta t \quad (7)$$

and the initial conditions are $x_0 = X$ and $v_0 = V$.

Recall that Euler's method has its drawbacks. One should know that the method is unstable for some problems, leading to fast increasing numerical errors. This, however, does not occur for the type of problem that is given below.

2 Cannon ball trajectory

The task is to simulate the trajectory of a cannon ball, including the effect of air resistance. The forces that act on the ball are illustrated in FIGURE 1, below.

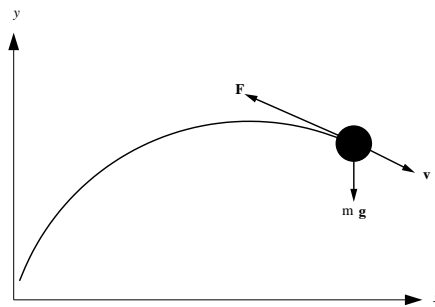


FIGURE 1. *Illustration of the trajectory, the velocity and the forces acting on the cannon ball. Gravity acts in the negative y -direction and the friction force is directed oppositely to the direction of motion.*

The forces are the gravity force, $m\mathbf{g}$, and the friction force, \mathbf{F} , due to the air. The ball's mass is denoted by m . Bold symbols denote vector quantities, e.g. $\mathbf{F} = (F_x, F_y)$. Since the motion is restricted to a plane (the xy -plane), we ignore the third spatial dimension. The equations of motion follow from Newton's law, $m\mathbf{a} = \mathbf{F}_{tot}$, where \mathbf{a} is the acceleration and \mathbf{F}_{tot} is the sum of the forces acting on the ball. In the described situation, the equations of motion can be written, in terms of the velocity \mathbf{v} , as

$$m \frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{F}$$

The position of the ball is given by the position vector $\mathbf{x} = (x, y)$. The velocity vector is defined $\mathbf{v} = \frac{d\mathbf{x}}{dt}$. Coordinates are chosen so that gravity acts in the negative y -direction, i.e. $\mathbf{g} = (0, -g)$. If the velocity of the cannon ball is not very high, the friction force is given by

$$\mathbf{F} = -m\kappa v\mathbf{v}$$

where v stands for the magnitude of the velocity, $v = (v_x^2 + v_y^2)^{1/2}$. Note that the air resistance gives a force \mathbf{F} that is *oppositely* directed to the velocity, see Fig. 1. The value of the constant κ depends on the mass density of the air,

ρ , the cannon ball's cross-section area, A , and on an air resistance coefficient, C , which is determined by the cannon ball's surface structure. These are related by $\kappa = \rho AC/(2m)$. If the trajectory is very high one should take into consideration that the air density varies with height. In this case κ becomes a function of y . From a simple model of the atmosphere, it follows that the air density as a function of height is given by $\rho(y) = \rho_0 e^{-y/L}$, with $L \approx 8$ km and ρ_0 denoting the mass density of air at sea level, i.e. $\rho_0 = \rho(0)$. The function $\kappa(y)$ can then be written $\kappa(y) = \kappa(0)e^{-y/L}$, where $\kappa(0) = \rho(0)AC/(2m)$.

The trajectory of a cannon ball, including the effect of air resistance, is thus given by the following equations, now written on component form as a set of first order differential equations:

$$\begin{aligned}\frac{dx}{dt} &= v_x \\ \frac{dy}{dt} &= v_y \\ \frac{dv_x}{dt} &= -\kappa(y)vv_x \\ \frac{dv_y}{dt} &= -g - \kappa(y)vv_y\end{aligned}$$

Use the following typical parameter values: $g = 9.82$ m/s², $\kappa(0) = 2 \cdot 10^{-4}$ m⁻¹ and $L = 8000$ m.

Exercise

- a) Write a MATLAB function that simulates the trajectory of a cannon ball using Euler's method. Input arguments should be the velocity v and initial angle $\alpha = \arctan(v_y(0)/v_x(0))$. The ground is assumed flat and the program should end when the ball hits the ground. Test the code by running it also for the case of no air resistance, for which you know the exact solution.
- b) For at least 7 initial velocities between 0 and 1700 m/s, determine those initial angles, α , that maximizes the length, x_{\max} , of the cannon ball trajectory (it is difficult to reach initial velocities larger than 1700 m/s with current technology). It is a good idea to let the computer find the maximizing angle in some systematic way. The trajectories should be illustrated graphically as well as the maximizing angle as a function of initial velocity.
- c) Assume that the air density *does not* depend on the height. How does this affect the length of the trajectories?

3 Report

Write a complete report. Note that a well written report will be very useful for you in the next course in Classical mechanics. In that course you will be assigned exercises where you should perform similar numerical computations. A well written report and numerical code from this laboration will save you a lot of time.

The report should have the following structure (with minor variations)

- Title page (title, name, date, course, programme)
- Abstract (summarizes the most important contents of the report)
- Table of contents
- Introduction
- Main part: method, algorithm, approximations (mathematical and physical), results
- Conclusions and discussion
- References (if needed)
- Appendix: longer analytical calculations, listing of the program code, data and figures that do not fit well into the main part

The report should be easy to understand for someone with the same background as yourself. Be careful to check the following:

- The number of significant digits.
- Units.
- Labels on the axes of figures.
- Figure and table captions. (Write a short explaining figure text to each figure. The *main* text should also describe and refer to the figures.)
- And don't forget to comment your code well!