5DV022 Problem Exercise 6 Fall 2008

Solutions to this exercise are due, at 1600 (4pm), on October 17, 2008. The signed cover sheet must be submitted with your solution.

In the course, it has been shown that by using the FFT as a tool in support of the multiplication of two polynomials of degree n, a time complexity of $\Theta(n \cdot \log(n))$ may be realized, which is a significant improvement over the complexity $\Theta(n^2)$ associated with the "naïve" algorithm. However, there are other algorithms which improve upon the $\Theta(n^2)$ bound, and in this exercise one of them will be investigated.

1. Let $p := a_0 + a_1x$ and $q := b_0 + b_1x$ be two polynomials of degree one. Show that the product

$$p \cdot q := a_0b_0 + (a_0b_1 + a_1b_0)x + a_1b_1x^2$$

may be realized with only three multiplications.
(Hint: As one of the multiplications, use $(a_0 + a_1) \cdot (b_0 + b_1)$).

2. Design a divide-and-conquer algorithm with time complexity $\Theta(n^{\log_2(3)})$ for the multiplication of two polynomials of degree $n-1$, based upon the technique which you developed in the solution to Problem 1. Sketch your algorithm clearly in high-level form. Additionally, show in step-by-step fashion how it multiplies the polynomials

$$\begin{aligned} p &:= a_0 + a_1x + a_2x^2 + a_3x^3 \\ q &:= b_0 + b_1x + b_2x^2 + b_3x^3 \end{aligned}$$

Your answer will be graded in large part based upon this step-through, so make sure that it is clear and complete.
Note: Your algorithm may assume that n is a power of 2.

3. Prove that your algorithm of Part 2 has worst-case time complexity $\Theta(n^{\log_2(3)})$, by writing down and then solving the associated recurrence relation.

4. Determine which complexity is lower, $\Theta(n \cdot \log(n))$ or $\Theta(n^{\log_2(3)})$. Prove your answer, do not just state it.

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(Hint: For Problem 4, you may find *l'Hôpital's rule* handy: If $f(x)$ and $g(x)$ are differentiable functions for which

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

is indeterminate; *i.e.*, either both of

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to\infty} g(x) = 0$$

hold, or else both of

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to\infty} g(x) = \infty$$

hold, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{df(x)/dx}{dg(x)/dx}$$

holds as well.