Orbit and bundle stratification of controllability and observability matrix pairs in StratiGraph

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State-space model and system pencil

A state-space system $\mathcal{S}$ with the *state-space model*

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

can be represented in the form of a *system pencil*

$$\mathbf{S}(\lambda) = \mathbf{A} - \lambda\mathbf{B} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},$$

with the corresponding general *matrix pencil* $\mathbf{A} - \lambda\mathbf{B}$.

In short form $\mathcal{S}$ is represented by a *matrix quadruple* (A, B, C, D).

Matrix pairs

We consider the subsystems of $\mathcal{S}$ corresponding to the *controllability pair* (A, B) and the *observability pair* (A, C), associated with

$$\dot{x}(t) = Ax(t) + Bu(t),$$

and

$$\dot{x}(t) = Ax(t),$$
$$y(t) = Cx(t),$$

respectively.

Their system pencil representations are

$$\mathbf{S}_{\mathrm{C}}(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_{\mathrm{O}}(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Outline

- State-space and matrix pairs
- Canonical forms
- Stratifications
- Orbits and bundles
- Integer partitions
- Main theorem: Covering relations for matrix pairs
- An example and StratiGraph

Canonical forms – Kronecker

Any matrix pencil $\mathbf{A} - \lambda\mathbf{B}$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations (U and V non-singular):

$$U^{-1}(\mathbf{A} - \lambda\mathbf{B})V =$$
$$\mathrm{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), N_{s_1}, \ldots, N_{s_k}, L^T_{\eta_1}, \ldots, L^T_{\eta_q}).$$

Singular part:

- $L_{\epsilon_1}, \ldots, L_{\epsilon_p}$ – Right singular blocks, ϵ_i are the *right minimal indices*.
- $L^T_{\eta_1}, \ldots, L^T_{\eta_q}$ – Left singular blocks, η_j are the *left minimal indices*.

Regular part:

- $J(\mu_1), \ldots, J(\mu_t)$ – Each $J(\mu_i)$ is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue μ_i.
- $N_{s_1}, \ldots, N_{s_k}$ – Jordan blocks corresponding to the inf. eigenvalue.

Canonical forms – canonical blocks

$$J_j(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i - \lambda \end{bmatrix}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix},$$

$$L_\epsilon = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad L_\eta^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix},$$

where $J_j(\mu_i)$ and N_j are of size $j \times j$, L_ϵ of size $\epsilon \times (\epsilon + 1)$ and L_η^T of size $(\eta + 1) \times \eta$.

Canonical forms – matrix pairs

- $\mathbf{S}_{\mathrm{C}}(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix}$ has *full row rank* $\Rightarrow$

  KCF of $\mathbf{S}_{\mathrm{C}}(\lambda)$ can only have finite eigenvalues (uncontrollable modes) and L_k blocks:

$$U^{-1}\mathbf{S}_{\mathrm{C}}(\lambda)V = \mathrm{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t)).$$

- $\mathbf{S}_{\mathrm{O}}(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ has *full column rank* $\Rightarrow$

  KCF of $\mathbf{S}_{\mathrm{O}}(\lambda)$ can only have finite eigenvalues (unobservable modes) and L_k^T blocks:

$$U^{-1}\mathbf{S}_{\mathrm{O}}(\lambda)V = \mathrm{diag}(J(\mu_1), \ldots, J(\mu_t), L_{\eta_1}^T, \ldots, L_{\eta_p}^T).$$

Stratifications – brief summary

An $n \times n$ matrix is a point in an n^2-dim (matrix) space.
Likewise, an $m \times n$ matrix pencil is a point in a $2mn$-dim space.

The set of all matrices with the same canonical form is a *manifold* in the n^2-dim space.
— Which other structures are in the closure of one such manifold?

Numerical computations — moving from point to point or manifold to manifold.
Manifolds — orbits of similar matrices, equivalent matrix pencils, etc.

A *stratification* is the closure hierarchy of orbits or bundles of canonical structures.

A structure *covers* another if its closure includes the closure of the other and there is no structures in between.

Orbits and bundles

Orbit of a matrix pencil:

$$\mathcal{O}(\mathbf{A} - \lambda\mathbf{B}) = \{U^{-1}(\mathbf{A} - \lambda\mathbf{B})V : \det(U)\det(V) \neq 0\}$$

Orbit of a controllability pair:

$$\mathcal{O}(A, B) = \left\{ P\,\mathbf{S}_{\mathrm{C}}(\lambda) \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det(P)\det(Q) \neq 0 \right\}$$

Orbit of an observability pair:

$$\mathcal{O}(A, C) = \left\{ \begin{bmatrix} P & S \\ 0 & T \end{bmatrix} \mathbf{S}_{\mathrm{O}}(\lambda) P^{-1} : \det(P)\det(T) \neq 0 \right\}$$

Bundle, $\mathcal{B}(*)$ — the union of orbits with unspecified eigenvalues.

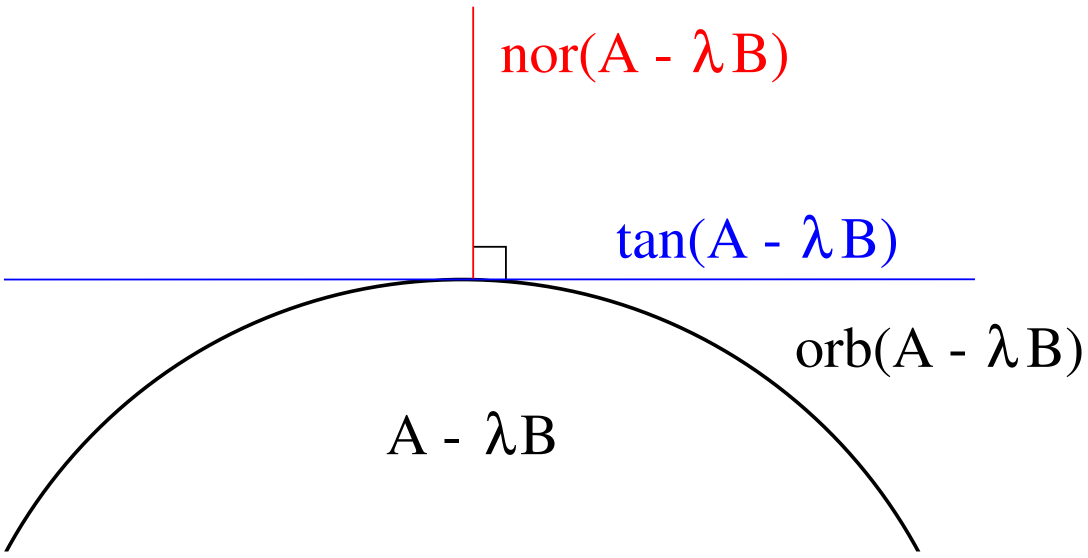
Normal space of orbit

Normal space, $\mathrm{nor}(\mathbf{A} - \lambda\mathbf{B})$ — The space complementary and orthogonal to $\tan(\mathbf{A} - \lambda\mathbf{B})$.

$$\dim(\tan(\mathbf{A} - \lambda\mathbf{B})) \equiv \dim(\mathcal{O}(\mathbf{A} - \lambda\mathbf{B}))$$

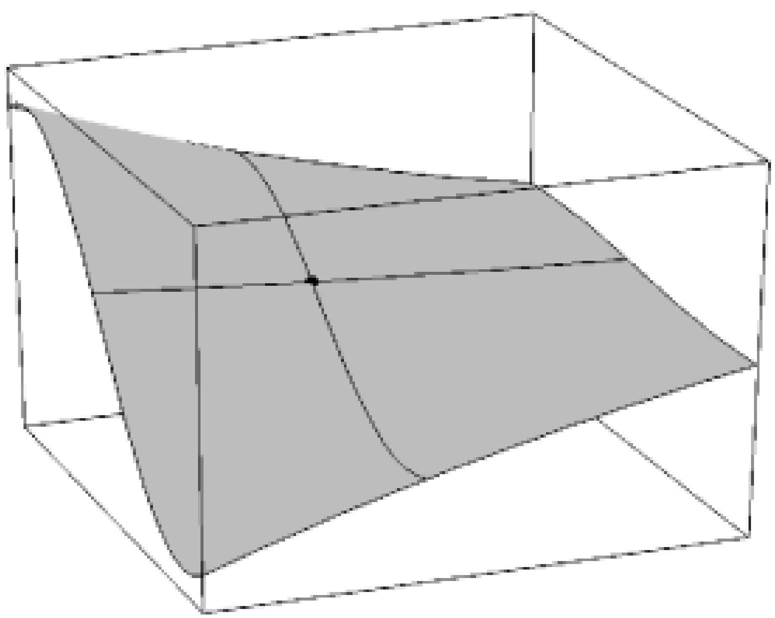
$$\dim(\tan(\mathbf{A} - \lambda\mathbf{B})) + \dim(\mathrm{nor}(\mathbf{A} - \lambda\mathbf{B})) = 2mn$$

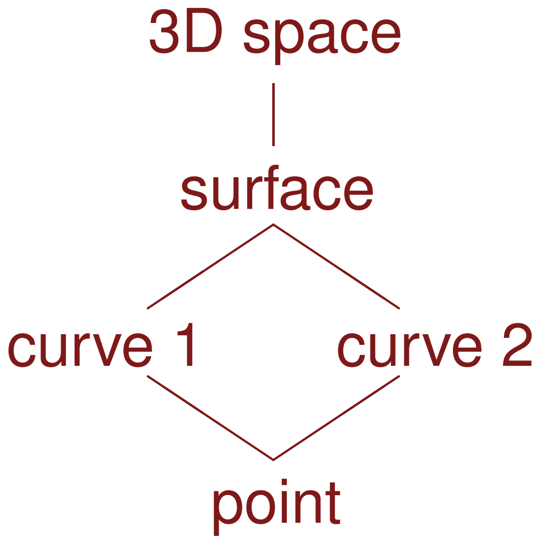
Codimension — Dimension of the normal space to $\mathcal{O}(\mathbf{A} - \lambda\mathbf{B})$.



Stratifications – graph representation

Illustration of closure hierarchy.

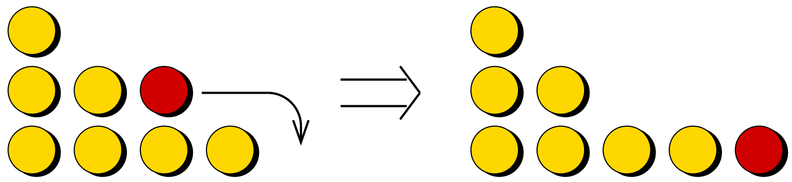




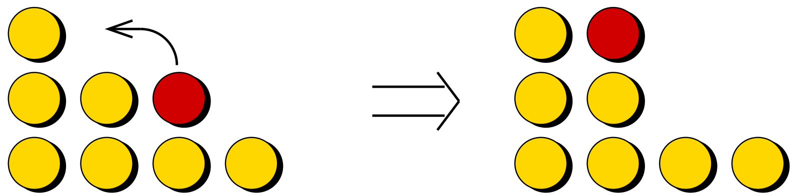
Integer partitions – tools for KCF repr.

A *partition* κ of an integer K is defined as $\kappa = (\kappa_1, \kappa_2, \dots)$ where $\kappa_1 \geq \kappa_2 \geq \cdots \geq 0$ and $K = \kappa_1 + \kappa_2 + \dots$.

Minimum rightward coin move: rightward *one* col or downward *one* row (keep partition monotonic).



Minimum leftward coin move: leftward *one* col or upward *one* row (keep partition monotonic).



KCF represented by integer partitions

- $\mathcal{J}_{\mu_i} = (j_1, j_2, \ldots)$ where $j_i = \#J_k(\mu_i)$ blocks with $k \geq i$. $\mathcal{J}_{\mu_i}$ is known as the *Weyr characteristics* of the finite eigenvalue μ_i.
- $\mathcal{N} = (n_1, n_2, \ldots)$ where $n_i = \#N_k$ with $k \geq i$. $\mathcal{N}$ is known as the Weyr characteristics of the infinite eigenvalue.
- $\mathcal{R} = (r_0, r_1, \ldots)$ where $r_i = \#L_k$ blocks with $k \geq i$.
- $\mathcal{L} = (l_0, l_1, \ldots)$ where $l_i = \#L_k^T$ blocks with $k \geq i$.

These block sizes are computed by staircase-type algorithms (e.g. GUPTRI (Demmel and Kågström)).

Thm: Covering relations for matrix pairs

Given the structure integer partitions $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of (A, B) or (A, C), the following if-and-only-if rules find $(\widetilde{A}, \widetilde{B})$ or $(\widetilde{A}, \widetilde{C})$ fulfilling orbit or bundle covering relations with (A, B) or (A, C), respectively.

$\mathcal{O}(A, B)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{B})$
(or $\mathcal{O}(A, C)$ covers $\mathcal{O}(\widetilde{A}, \widetilde{C})$):

(1) Minimum *rightward* coin move in $\mathcal{R}$ (or $\mathcal{L}$).

(2) If the rightmost column in $\mathcal{R}$ (or $\mathcal{L}$) is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially).

(3) Minimum *leftward* coin move in any $\mathcal{J}_{\mu_i}$.

Rules 1 and 2 may not make coin moves that affect r_0 (or l_0).

$\mathcal{O}(A, B)$ is covered by $\mathcal{O}(\widetilde{A}, \widetilde{B})$
(or $\mathcal{O}(A, C)$ is covered by $\mathcal{O}(\widetilde{A}, \widetilde{C})$):

(1) Minimum *leftward* coin move in $\mathcal{R}$ (or $\mathcal{L}$), without affecting r_0 (or l_0).

(2) If the rightmost column in some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$ (or $\mathcal{L}$).

(3) Minimum *rightward* coin move in any $\mathcal{J}_{\mu_i}$.

Similar theorem exists for the bundle case.

Example

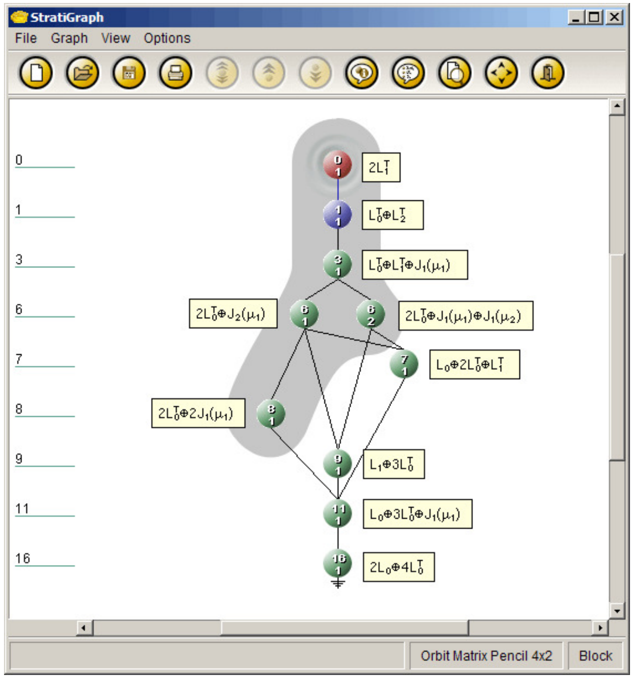
Consider the system pencil

$$\mathbf{S}(\lambda) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A,\ B,\ C \in \mathbb{C}^{2\times 2}$.

$\mathrm{S}(\lambda)$ as a general 4×4 matrix pencil
$\Rightarrow$ 47 structures!
$\mathrm{S}_{\mathrm{O}}(\lambda)$ as a general 4×2 matrix pencil
$\Rightarrow$ 10 structures!

Does not take advantage of the special structure of $\mathrm{S}(\lambda)$ or $\mathrm{S}_{\mathrm{O}}(\lambda)$.



Example – stratification

$\mathcal{O}(A, C)$

L_2^T

Example – stratification

$\mathcal{O}(A, C)$

$$\begin{array}{c} L_2^T \\ | \\ L_1^T \oplus J_1(\alpha) \end{array}$$

(2) If the rightmost column in $\mathcal{R}$ (or $\mathcal{L}$) is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially).
The rule may not make coin moves that affect r_0 (or l_0).

$$\begin{array}{lcl} \mathcal{L} = (1, 1, 1) & \Rightarrow & \mathcal{L} = (1, 1) \\ \mathcal{J}_\alpha = () & \Rightarrow & \mathcal{J}_\alpha = (1) \end{array}$$

Example – stratification

$$\mathcal{O}(A, C)$$

$$L_2^T$$

$$L_1^T \oplus J_1(\alpha)$$

$$L_0^T \oplus J_2(\alpha)$$

(2) Move the single rightmost coin in $\mathcal{L}$ to a new column in the *existing* $\mathcal{J}_\alpha$.

$$\mathcal{L} = (1, 1) \quad \Rightarrow \quad \mathcal{L} = (1)$$

$$\mathcal{J}_\alpha = (1) \quad \Rightarrow \quad \mathcal{J}_\alpha = (1, 1)$$

Example – stratification

$\mathcal{O}(A, C)$

L_2^T

$L_1^T \oplus J_1(\alpha)$

$L_0^T \oplus J_2(\alpha)$ $\quad$ $L_0^T \oplus J_1(\alpha) \oplus J_1(\beta)$

(2) Move the single rightmost coin in $\mathcal{L}$ to a *new* partition $\mathcal{J}_\beta$.

$$\begin{aligned}
\mathcal{L} &= (1, 1) & \Rightarrow \quad \mathcal{L} &= (1) \\
\mathcal{J}_\alpha &= (1) & \Rightarrow \quad \mathcal{J}_\alpha &= (1) \\
\mathcal{J}_\beta &= () & \Rightarrow \quad \mathcal{J}_\beta &= (1)
\end{aligned}$$

Example – stratification

$\mathcal{O}(A, C)$

$$
\begin{tikzcd}
 & L_2^T \arrow[d, dash] & \\
 & L_1^T \oplus J_1(\alpha) \arrow[dl, dash] \arrow[dr, dash] & \\
L_0^T \oplus J_2(\alpha) \arrow[dr, dash] & & L_0^T \oplus J_1(\alpha) \oplus J_1(\beta) \\
 & L_0^T \oplus 2J_1(\alpha) &
\end{tikzcd}
$$

(3) Minimum *leftward* coin move in any $\mathcal{J}_{\mu_i}$.

$$
\begin{array}{lcl}
\mathcal{L} = (1) & \Rightarrow & \mathcal{L} = (1) \\
\mathcal{J}_\alpha = (1, 1) & \Rightarrow & \mathcal{J}_\alpha = (2)
\end{array}
$$

Example – stratification

$\mathcal{O}(A, C)$

$$
\begin{tikzcd}[column sep=small]
 & L_2^T \arrow[d, dash] & \\
 & L_1^T \oplus J_1(\alpha) \arrow[dl, dash] \arrow[dr, dash] & \\
L_0^T \oplus J_2(\alpha) \arrow[dr, dash] & & L_0^T \oplus J_1(\alpha) \oplus J_1(\beta) \\
 & L_0^T \oplus 2J_1(\alpha) &
\end{tikzcd}
$$

$\mathcal{O}(A, B)$

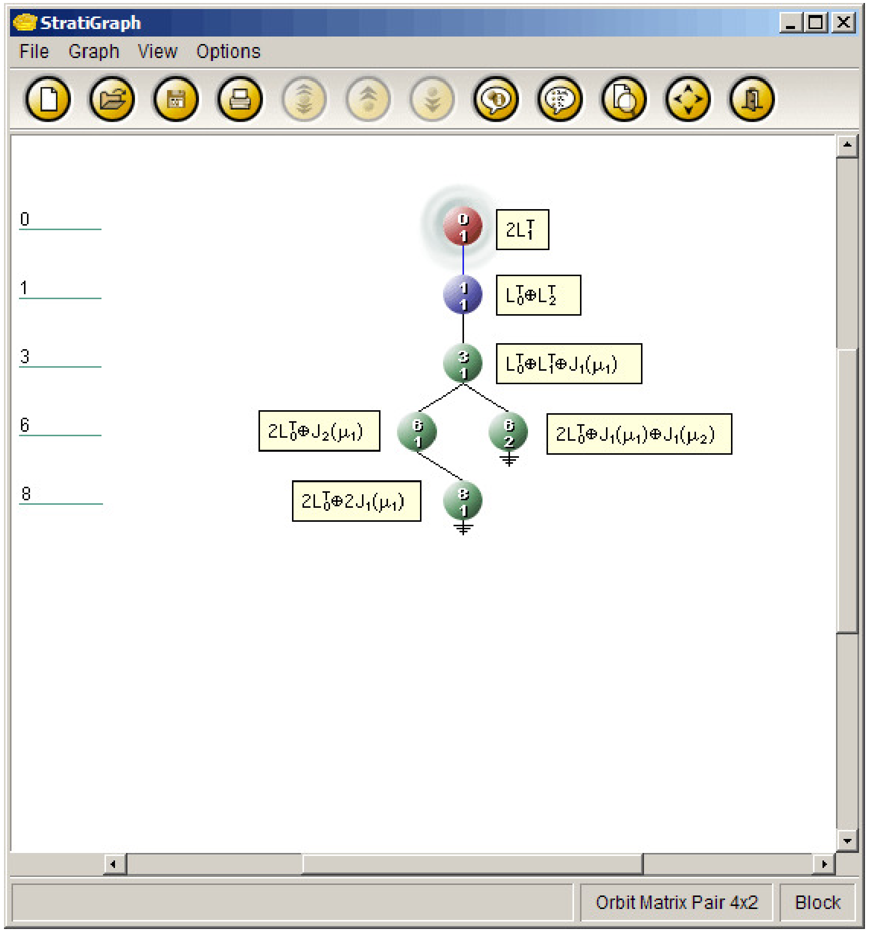
$$
\begin{tikzcd}[column sep=small]
 & L_2 \arrow[d, dash] & \\
 & L_1 \oplus J_1(\alpha) \arrow[dl, dash] \arrow[dr, dash] & \\
L_0 \oplus J_2(\alpha) \arrow[dr, dash] & & L_0 \oplus J_1(\alpha) \oplus J_1(\beta) \\
 & L_0 \oplus 2J_1(\alpha) &
\end{tikzcd}
$$

StratiGraph

StratiGraph is a software tool for computing and visualizing the closure hierarchy of orbits and bundles.

Support for matrices, matrix pencils and matrix pairs (both controllability and observability pairs).

Developed at Department of Computing Science, Umeå University, by Pedher Johansson (in collaboration with Bo Kågström and Erik Elmroth).



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To download StratiGraph and for more information visit:

http://www.cs.umu.se/research/nla/singular_pairs/