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Computing Periodic Deflating Subspaces Associated with a Specified Set of Eigenvalues *†

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Abstract.

We present a direct method for reordering eigenvalues in the generalized periodic real Schur form of a regular K-cylic matrix pair sequence (A_k, E_k). Following and generalizing existing approaches, reordering consists of consecutively computing the solution to an associated Sylvester-like equation and constructing K pairs of orthogonal matrices. These pairs define an orthogonal K-cyclic equivalence transformation that swaps adjacent diagonal blocks in the Schur form. An error analysis of this swapping procedure is presented, which extends existing results for reordering eigenvalues in the generalized real Schur form of a regular pair (A, E). Our direct reordering method is used to compute periodic deflating subspace pairs corresponding to a specified set of eigenvalues. This computational task arises in various applications related to discrete-time periodic descriptor systems. Computational experiments confirm the stability and reliability of the presented eigenvalue reordering method.

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Key words: generalized product of a K-cyclic matrix pair sequence, generalized periodic real Schur form, eigenvalue reordering, periodic generalized coupled Sylvester equation, K-cyclic equivalence transformation, generalized periodic eigenvalue problem.

1 Introduction

Discrete-time periodic descriptor systems of the form

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k + D_k u_k, \end{aligned} \tag{1.1}$$

with $A_k = A_{k+K}$, $E_k = E_{k+K} \in \mathbb{R}^{n\times n}$, $B_k = B_{k+K} \in \mathbb{R}^{n\times m}$, $C_k = C_{k+K} \in \mathbb{R}^{r\times n}$ and $D_k = D_{k+K} \in \mathbb{R}^{r\times m}$ for some period $K \geq 1$ arise naturally from processes that exhibit seasonal or periodic behavior [6]. Design and analysis

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problems of such systems (see, e.g., [30, 31, 38]) are conceptually studied in terms of *state transition matrices* [38] $\Phi_{E^{-1}A}(j,i) = E_{j-1}^{-1}A_{j-1}E_{j-2}^{-1}A_{j-2}\dots E_i^{-1}A_i \in \mathbb{R}^{n\times n}$, with the convention $\Phi_{E^{-1}A}(i,i) = I_n$. A state transition matrix over a complete period $\Phi_{E^{-1}A}(j+K,j)$ is the *monodromy matrix* of (1.1) at time j. Its eigenvalues are called the *characteristic multipliers* and are independent of the time j. Specifically, the monodromy matrix at time $j = 0$ corresponds to the matrix product

$$E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_1^{-1}A_1E_0^{-1}A_0. \tag{1.2}$$

Matrix products of the general form (1.2) are studied, e.g., in [3, 5, 25, 39].

We study the K-cyclic matrix pair sequence (A_k, E_k) with $A_k, E_k \in \mathbb{R}^{n\times n}$ from (1.1) via the generalized periodic Schur decomposition [8, 17]: there exists a K-cyclic orthogonal matrix pair sequence (Q_k, Z_k) with $Q_k, Z_k \in \mathbb{R}^{n\times n}$ such that, given $k \oplus 1 = (k+1) \bmod K$, we have

$$\left\{ \begin{array}{rcl} S_k &=& Q_k^T A_k Z_k, \\ T_k &=& Q_k^T E_k Z_{k\oplus 1}, \end{array} \right. \tag{1.3}$$

where all matrices S_k, except for some fixed index j with $0 \le j \le K-1$, and all matrices T_k are upper triangular. The matrix S_j is upper quasi-triangular; typically j is chosen to be 0 or $K-1$. The sequence (S_k, T_k) is the *generalized periodic real Schur form* (GPRSF) of (A_k, E_k), $k = 0, 1, \dots, K-1$. The decomposition (1.3) is a *K-cyclic equivalence transformation* of the matrix pair sequence (A_k, E_k).

Computing the GPRSF is the standard method for solving the *generalized periodic (product) eigenvalue problem* (GPEVP)

$$E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_1^{-1}A_1E_0^{-1}A_0x = \lambda x, \tag{1.4}$$

where all matrices in the pairs (A_k, E_k) are general and dense. For $K = 1$, (1.4) corresponds to a generalized eigenvalue problem $Ax = \lambda Ex$ with (A, E) regular (see, e.g., [12]). Using the GPRSF to solve a GPEVP for $K \ge 1$ means that we do not need to compute any matrix products in (1.4) explicitly, which avoids numerical instabilities and allows to handle singular factors E_k.

The 1×1 and 2×2 blocks on the diagonal of a GPRSF define $t \le n$ K-cyclic diagonal block pairs $(S_{ii}^{(k)}, T_{ii}^{(k)})$, corresponding to *real eigenvalues* and *complex conjugate pairs of eigenvalues*, respectively.

A real eigenvalue is simply given by

$$\lambda_i = \prod_{k=K-1}^{0} \frac{S_{ii}^{(k)}}{T_{ii}^{(k)}}.$$

This eigenvalue is called *infinite* if $\prod_{k=K-1}^{0} T_{ii}^{(k)} = 0$ but $\prod_{k=K-1}^{0} S_{ii}^{(k)} \neq 0$. If there are 1×1 blocks for which both $\prod_{k=K-1}^{0} S_{ii}^{(k)} = 0$ and $\prod_{k=K-1}^{0} T_{ii}^{(k)} = 0$ then the K-cyclic matrix pair sequence (A_k, E_k) is called *singular*, otherwise

the sequence (A_k, E_k) is called *regular*. In the degenerate singular case, the eigenvalues become ill-defined and other tools [27, 36] need to be used to study the periodic eigenvalue problem. For the rest of the paper, it is therefore assumed that (A_k, E_k) is regular.

For two complex conjugate eigenvalues $\lambda_i, \bar{\lambda}_i$, all matrices $T_{ii}^{(k)}$ are nonsingular and

$$\lambda_i, \bar{\lambda}_i \in \lambda \left(\prod_{k=K-1}^{0} {T_{ii}^{(k)}}^{-1} S_{ii}^{(k)} \right),$$

where $\lambda(M)$ denotes the set of eigenvalues of a matrix M. In finite precision arithmetic, great care has to be exercised to avoid underflow and overflow in the explicit eigenvalue computation involving 2×2 blocks [34].

For every l with $1 \leq l \leq n$ such that no 2×2 block resides in $S_j(l{:}l{+}1, l{:}l{+}1)$, the first l pairs of columns of (Q_0, Z_0) span a *deflating subspace pair* corresponding to the first l eigenvalues of the matrix product (1.2). More generally, the first l pairs of columns of (Q_k, Z_k) span a *left and right periodic (or cyclic) deflating subspace pair* sequence associated with the first l eigenvalues of the matrix product (1.2) [5].

The decomposition (1.3) is computed via the periodic QZ algorithm (see, e.g., [8, 17, 23, 24]), which consists of an initial reduction to generalized periodic Hessenberg form and a subsequent iterative process to generalized periodic Schur form. In [37], the generalized periodic Schur form is extended to periodic matrix pairs with time-varying and possibly rectangular dimensions. This includes a preprocessing step that truncates parts corresponding to spurious characteristic values, which then yields square system matrices of constant dimensions.

1.1 Ordered GPRSF and periodic deflating subspaces

In many applications, it is desirable to have the eigenvalues along the diagonal of the GPRSF in a certain order. If the generalized periodic Schur form has its eigenvalues ordered in a certain way as in (1.5), it is called an *ordered GPRSF*. For example, if we have

$$S_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix}, \quad T_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix}, \tag{1.5}$$

with $S_{11}^{(k)}, T_{11}^{(k)} \in \mathbb{R}^{l \times l}$ such that the upper left part sequence $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues in the open unit disc, then (S_k, T_k) is an ordered GPRSF and the first l columns of the sequence Z_k span *stable* right periodic deflating subspaces. For initial states $x_0 \in \operatorname{span}(Z_0 e_1, \ldots, Z_0 e_l)$ with e_i being the ith unit vector, the states of the open loop system $E_k x_{k+1} = A_k x_k$ satisfy $x_k \in \operatorname{span}(Z_k e_1, \ldots, Z_k e_l)$ and 0 is an asymptotically stable equilibrium.

Other important applications relating to periodic discrete-time systems include the *stable-unstable spectral separation* for computing the numerical solution of the discrete-time periodic Riccati equation [37] in LQ-optimal control, which we illustrate in Section 2, and pole placement where the goal is to move some or all

of the poles to desired locations in the complex plane [28, 15]. In [4], ordered Schur forms are used for solving generalized Hamiltonian eigenvalue problems.

In this paper, we extend the work in [2, 14, 20, 24, 15] to perform eigenvalue reordering in a regular periodic matrix pair sequence in GPRSF.

The rest of the paper is organized as follows. In Section 2, we illustrate how an ordered GPRSF can be used to solve the discrete-time periodic Riccati equation that arises in an LQ-optimal control problem. Section 3 presents our direct method for reordering eigenvalues of a periodic (cyclic) matrix pair sequence (A_k, E_k) in GPRSF. To compute an ordered GPRSF, a method for reordering adjacent K-cyclic diagonal block pairs is combined with a bubble-sort like procedure in an LAPACK-style [1, 2, 22] fashion. The proposed method for swapping adjacent diagonal block pair sequences relies on orthogonal K-cyclic equivalence transformations and the core step consists of computing the solution to an associated periodic generalized coupled Sylvester equation, which is discussed in Section 4. An error analysis of the direct reordering method is presented in Section 5, which extends and generalizes results from [20, 14]. In Section 6, we discuss some implementation issues regarding the solution of small-sized periodic generalized coupled Sylvester equations and how we control and guarantee stability of the reordering. Some examples and computational results are presented and discussed in Section 7. Finally, in Section 8 we discuss some extensions of the reordering method.

2 LQ-optimal control and periodic deflating subspaces

Given the system (1.1), the aim of *linear quadratic* (LQ) *optimal control* is to find a feedback sequence u_k which stabilizes the system and minimizes the functional

$$\frac{1}{2}\sum_{k=0}^{\infty}(x_k^T H_k x_k + u_k^T N_k u_k),$$

with $H_k \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite and $N_k \in \mathbb{R}^{m\times m}$ symmetric positive definite. Moreover, we suppose that the weighting matrices are K-periodic, i.e., $H_{k+K} = H_k$ and $N_{k+K} = N_k$. Under mild assumptions [7], the optimal feedback is linear and unique. For each k, it can be expressed as

$$u_k^\star = -(N_k + B_k^T X_{k+1} B_k)^{-1} B_k^T X_{k+1} A_k x_k,$$

where $X_k = X_{k+K}$ is the unique symmetric positive semidefinite solution of the *discrete-time periodic Riccati equation* (DPRE) [17]

$$\begin{aligned} 0 = {} & C_k^T H_k C_k - E_{k-1}^T X_k E_{k-1} + A_k^T X_{k+1} A_k \\ & - A_k^T X_{k+1} B_k (N_k + B_k^T X_{k+1} B_k)^{-1} B_k^T X_{k+1} A_k, \end{aligned} \tag{2.1}$$

provided that all E_k are invertible. The $2n \times 2n$ periodic matrix pair

$$(L_k, M_k) = \left(\begin{bmatrix} A_k & 0 \\ -C_k^T H_k C_k & E_{k-1}^T \end{bmatrix}, \begin{bmatrix} E_{k-1} & B_k N_k^{-1} B_k^T \\ 0 & A_k^T \end{bmatrix} \right)$$

is closely associated with (2.1). Similarly as for the case $E_k = I_n$ [17], it can be shown that this pair has exactly n eigenvalues inside the unit disk under the assumption that (1.1) is *d-stabilizable* and *d-detectable*. By reordering the GPRSF of (L_k, M_k) we can compute a periodic deflating subspace defined by the orthogonal matrices $U_k, V_k \in \mathbb{R}^{2n\times 2n}$ with $V_{k+K} = V_k$ such that

$$U_k^T L_k V_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix},\ U_k^T M_k V_{k+1} = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix},$$

where the $n \times n$ periodic matrix pair $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues inside the unit disk. If we partition

$$U_k = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}$$

with $U_{ij}^{(k)} \in \mathbb{R}^{n\times n}$, then

$$U_{21}^{(k)} \left[U_{11}^{(k)}\right]^{-1} = X_k E_{k-1},$$

from which X_k can be computed. The proof of this relation is similar as for the case $K = 1$, see, e.g., [26]. We note that if N_k is not well-conditioned then it is better to work with $3n \times 3n$ matrix pairs, as described in [26].

3 Direct method for eigenvalue reordering in GPRSF

Given a regular K-cyclic matrix pair sequence (A_k, E_k) in GPRSF, our method to compute an ordered GPRSF (1.5) with respect to a set of specified eigenvalues reorders 1×1 and 2×2 diagonal blocks in the GPRSF such that the selected set of eigenvalues appears in the matrix pair sequence $(S_{11}^{(k)}, T_{11}^{(k)})$. Following LAPACK, we assume that the set of specified eigenvalues are provided as an index vector for the blocks of eigenvalue pairs that should appear in $(S_{11}^{(k)}, T_{11}^{(k)})$. The procedure is now to swap adjacent diagonal blocks in the GPRSF in a bubble-sort fashion such that the specified eigenvalue ordering is satisfied [1, 2, 22]. In the following, we focus on the K-cyclic swapping of diagonal blocks using orthogonal transformations.

3.1 Swapping of K-cyclic diagonal block matrix pairs

Consider a regular K-cyclic matrix pair sequence (A_k, E_k) in GPRSF

$$(A_k, E_k) = \left(\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix}, \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} \right) \tag{3.1}$$

with $A_{11}^{(k)}, E_{11}^{(k)} \in \mathbb{R}^{p_1\times p_1}$ and $A_{22}^{(k)}, E_{22}^{(k)} \in \mathbb{R}^{p_2\times p_2}$, for $k = 0, 1, \ldots, K-1$.

Swapping consists of computing orthogonal matrices U_k, V_k such that

$$\begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} V_k, \tag{3.2}$$

$$\begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} V_{k\oplus 1}, \tag{3.3}$$

for $k = 0, \ldots, K-1$, and

$$\lambda(\hat{\Pi}_{11}) = \lambda(\Pi_{22}), \quad \lambda(\hat{\Pi}_{22}) = \lambda(\Pi_{11}), \tag{3.4}$$

where

$$\Pi_{ii} = \left[E_{ii}^{(K-1)}\right]^{-1} A_{ii}^{(K-1)} \cdots \left[E_{ii}^{(0)}\right]^{-1} A_{ii}^{(0)}, \tag{3.5}$$

$$\hat{\Pi}_{ii} = \left[\hat{E}_{ii}^{(K-1)}\right]^{-1} \hat{A}_{ii}^{(K-1)} \cdots \left[\hat{E}_{ii}^{(0)}\right]^{-1} \hat{A}_{ii}^{(0)}. \tag{3.6}$$

If some of the $E_{ii}^{(k)}$ are singular then the products (3.5) and (3.6) should only be understood in a formal sense, with their finite and infinite eigenvalues defined via the GPRSF. The relation (3.4) means that all eigenvalues of Π_{22} are transferred to $\hat{\Pi}_{11}$ and all eigenvalues of Π_{11} to $\hat{\Pi}_{22}$. For our purpose, $A_{ii}^{(k)}, E_{ii}^{(k)} \in \mathbb{R}^{p_i \times p_i}$ are the diagonal blocks of a GPRSF and it can thus be assumed that $p_i \in \{1, 2\}$.

The K-cyclic swapping is performed in two main steps. First, the sequence (A_k, E_k) in (3.1) is block diagonalized by a nonorthogonal K-cyclic equivalence transformation. Second, orthogonal transformation matrices are computed from this matrix pair sequence that perform the required K-cyclic swapping.

3.2 Swapping by block diagonalization and permutation

Let us consider a K-cyclic matrix pair sequence (L_k, R_k), with $L_k, R_k \in \mathbb{R}^{p_1 \times p_2}$, which solves the *periodic generalized coupled Sylvester equation* (PGCSY)

$$\begin{cases} A_{11}^{(k)} R_k - L_k A_{22}^{(k)} = -A_{12}^{(k)}, \\ E_{11}^{(k)} R_{k\oplus 1} - L_k E_{22}^{(k)} = -E_{12}^{(k)}. \end{cases} \tag{3.7}$$

Then (L_k, R_k) defines an equivalence transformation that block diagonalizes the K-cyclic matrix pair sequence (A_k, E_k) in (3.1):

$$\begin{aligned} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} &= \begin{bmatrix} I_{p_1} & L_k \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I_{p_1} & -R_k \\ 0 & I_{p_2} \end{bmatrix}, \\ \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} &= \begin{bmatrix} I_{p_1} & L_k \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} E_{11}^{(k)} & 0 \\ 0 & E_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I_{p_1} & -R_{k\oplus 1} \\ 0 & I_{p_2} \end{bmatrix}, \end{aligned} \tag{3.8}$$

for $k = 0, 1, \ldots, K-1$. In Section 4, we show that PGCSY (3.7) has a unique solution if and only if

$$\lambda(\Phi_{E_{11}^{-1}A_{11}}(K, 0)) \cap \lambda(\Phi_{E_{22}^{-1}A_{22}}(K, 0)) = \emptyset. \tag{3.9}$$

We refer to (L_0, R_0) as the *generator matrix pair* for the periodic reordering of $\Phi_{E^{-1}A}(K, 0)$ [14]. Periodic matrix Sylvester-type equations were studied in, e.g., [29, 35] and an overview was given in [38]. The PGCSY equation was recently studied in [13].

The diagonal blocks of the block diagonal matrices in (3.8) are swapped by a simple equivalence permutation:

$$\begin{aligned}\left[\begin{array}{cc} 0 & I_{p_2} \\ I_{p_1} & 0 \end{array}\right] \left(\left[\begin{array}{cc} A_{11}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{array}\right], \left[\begin{array}{cc} E_{11}^{(k)} & 0 \\ 0 & E_{22}^{(k)} \end{array}\right] \right) \left[\begin{array}{cc} 0 & I_{p_1} \\ I_{p_2} & 0 \end{array}\right] = \\ \left(\left[\begin{array}{cc} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{array}\right], \left[\begin{array}{cc} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{array}\right] \right).\end{aligned} \tag{3.10}$$

Altogether, by defining the matrices

$$\boldsymbol{X}_k = \left[\begin{array}{cc} L_k & I_{p_1} \\ I_{p_2} & 0 \end{array}\right], \quad \boldsymbol{Y}_k = \left[\begin{array}{cc} 0 & I_{p_2} \\ I_{p_1} & -R_k \end{array}\right], \quad k = 0, \ldots, K-1,$$

we obtain a *non-orthogonal* K-cyclic equivalence transformation such that

$$\begin{aligned}\left[\begin{array}{cc} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{array}\right] &= \boldsymbol{X}_k \left[\begin{array}{cc} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{array}\right] \boldsymbol{Y}_k, \\ \left[\begin{array}{cc} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{array}\right] &= \boldsymbol{X}_k \left[\begin{array}{cc} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{array}\right] \boldsymbol{Y}_{k\oplus 1}.\end{aligned} \tag{3.11}$$

3.3 Swapping by orthogonal transformation matrices

From the observation that the first block column of $\boldsymbol{X}_k$ and the last block row of $\boldsymbol{Y}_k$ have full column and row ranks, respectively, we can choose orthogonal matrices Q_k and Z_k from QR and RQ factorizations such that

$$\left[\begin{array}{c} L_k \\ I_{p_2} \end{array}\right] = Q_k \left[\begin{array}{c} T_L^{(k)} \\ 0 \end{array}\right], \quad \left[\begin{array}{cc} I_{p_1} & -R_k \end{array}\right] = \left[\begin{array}{cc} 0 & T_R^{(k)} \end{array}\right] Z_k^T, \tag{3.12}$$

where $T_L^{(k)} \in \mathbb{R}^{p_2 \times p_2}, T_R^{(k)} \in \mathbb{R}^{p_1 \times p_1}$ are all non-singular and upper triangular for $k = 0, 1, \ldots, K-1$.

Partitioning Q_k and Z_k in conformity with $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ as

$$Q_k = \left[\begin{array}{cc} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{array}\right], \quad Z_k = \left[\begin{array}{cc} Z_{11}^{(k)} & Z_{12}^{(k)} \\ Z_{21}^{(k)} & Z_{22}^{(k)} \end{array}\right],$$

we observe that

$$Q_k^T \boldsymbol{X}_k = \left[\begin{array}{cc} T_L^{(k)} & {Q_{12}^{(k)}}^T \\ 0 & {Q_{22}^{(k)}}^T \end{array}\right], \quad \boldsymbol{Y}_k Z_k = \left[\begin{array}{cc} Z_{21}^{(k)} & Z_{22}^{(k)} \\ 0 & T_R^{(k)} \end{array}\right]. \tag{3.13}$$

By applying (Q_k, Z_k) as an *orthogonal* K-cyclic equivalence transformation to (A_k, E_k) we obtain

$$(Q_k^T A_k Z_k, Q_k^T E_k Z_{k\oplus 1}) =$$

$$\left(Q_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} Z_k, Q_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} Z_{k\oplus 1}\right) =$$

$$\left(Q_k^T \boldsymbol{X}_k \begin{bmatrix} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_k Z_k, Q_k^T \boldsymbol{X_k} \begin{bmatrix} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_{k\oplus 1} Z_{k\oplus 1}\right) \equiv$$

$$\left(\begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix}, \begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix}\right),$$

where

$$\begin{cases} \hat{A}_{11}^{(k)} &= T_L^{(k)} A_{22}^{(k)} Z_{21}^{(k)}, \\ \hat{A}_{12}^{(k)} &= T_L^{(k)} A_{22}^{(k)} Z_{22}^{(k)} + {Q_{11}^{(k)}}^T A_{11}^{(k)} T_R^{(k)}, \\ \hat{A}_{22}^{(k)} &= {Q_{12}^{(k)}}^T A_{11}^{(k)} T_R^{(k)}, \end{cases} \tag{3.14}$$

and

$$\begin{cases} \hat{E}_{11}^{(k)} &= T_L^{(k)} E_{22}^{(k)} Z_{21}^{(k\oplus 1)}, \\ \hat{E}_{12}^{(k)} &= T_L^{(k)} E_{22}^{(k)} Z_{22}^{(k\oplus 1)} + {Q_{11}^{(k)}}^T E_{11}^{(k)} T_R^{(k\oplus 1)}, \\ \hat{E}_{22}^{(k)} &= {Q_{12}^{(k)}}^T E_{11}^{(k)} T_R^{(k\oplus 1)}. \end{cases} \tag{3.15}$$

From the equations above, we see that $(A_{11}^{(k)}, E_{11}^{(k)})$ and $(A_{22}^{(k)}, E_{22}^{(k)})$ are *K-cyclic equivalent* to $(\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ and $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)})$, respectively. In other words, the eigenvalues of the K-cyclic matrix pair sequence (A_k, E_k) have been reordered as desired.

We remark that $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)})$ and $(\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ are generally not in GPRSF after the K-cyclic swapping and have to be further transformed by orthogonal transformations to restore the GPRSF of the matrix pair sequence (A_k, E_k) (see Section 6.2).

4 The periodic generalized coupled Sylvester equation

The core step of the reordering method is to solve the associated PGCSY (3.7). Using Kronecker products this problem can be rewritten as a linear system of equations

$$Z_{\mathrm{PGCSY}} x = c, \tag{4.1}$$

where Z_{PGCSY} is a $2Kp_1p_2 \times 2Kp_1p_2$ matrix representation of the periodic generalized coupled Sylvester operator defined by the left hand sides of (3.7) for $k = 0, \ldots, K-1$:

$$Z_{\mathrm{PGCSY}} = \begin{bmatrix} -A_{22}^{(0)^T} \otimes I_{p_1} & & & & & & I_{p_2} \otimes A_{11}^{(0)} \\ -E_{22}^{(0)^T} \otimes I_{p_1} & I_{p_2} \otimes E_{11}^{(0)} & & & & & \\ & I_{p_2} \otimes A_{11}^{(1)} & -A_{22}^{(1)^T} \otimes I_{p_1} & & & & \\ & & -E_{22}^{(1)^T} \otimes I_{p_1} & \ddots & & & \\ & & & \ddots & & & \\ & & & & -A_{22}^{(K-1)^T} \otimes I_{p_1} & & \\ & & & & -E_{22}^{(K-1)^T} \otimes I_{p_1} & I_{p_2} \otimes E_{11}^{(K-1)} \end{bmatrix},$$

and x and c are $2Kp_1p_2 \times 1$ vector representations of the assembled unknowns and right hand sides, respectively:

$$x = \begin{bmatrix} \mathrm{vec}(L_0) \\ \mathrm{vec}(R_1) \\ \mathrm{vec}(L_1) \\ \mathrm{vec}(R_2) \\ \vdots \\ \mathrm{vec}(R_{K-1}) \\ \mathrm{vec}(L_{K-1}) \\ \mathrm{vec}(R_0) \end{bmatrix}, \qquad c = \begin{bmatrix} \mathrm{vec}(-A_{12}^{(0)}) \\ \mathrm{vec}(-E_{12}^{(0)}) \\ \mathrm{vec}(-A_{12}^{(1)}) \\ \mathrm{vec}(-E_{12}^{(1)}) \\ \vdots \\ \mathrm{vec}(-A_{12}^{(K-1)}) \\ \mathrm{vec}(-E_{12}^{(K-1)}) \end{bmatrix}.$$

Here, the operator $\mathrm{vec}(M)$ stacks the columns of a matrix M on top each other into one long vector. Note also that only the nonzero blocks of Z_{PGCSY} are displayed explicitly above. The sparsity structure of Z_{PGCSY} can be exploited when using Gaussian elimination with partial pivoting (GEPP) or a QR factorization to solve (4.1), see Section 6 for more details.

If all $E_{ii}^{(k)}$ are nonsingular, the PGCSY equation (3.7) can be manipulated to achieve a sequence of independent *generalized coupled Sylvester* (GCSY) *equations* as

$$(4.2)\begin{cases} A_{11}^{(k)} R_k - L_k A_{22}^{(k)} = -A_{12}^{(k)}, \\ A_{11}^{(k)} \Phi_{E_{11}^{-1}A_{11}}(K+k,k)^{-1} R_k - L_k A_{22}^{(k)} \Phi_{E_{22}^{-1}A_{22}}(K+k,k)^{-1} = C_k, \end{cases}$$

$k = 0, 1, \ldots, K-1$, where C_k is a sum of products formed from blocks of the left and right hand sides in (3.7). It is well-known that the GCSY (4.2) has a unique solution X_k for each k if and only if the regular matrix pairs

$$(A_{11}^{(k)}, A_{11}^{(k)} \Phi_{E_{11}^{-1}A_{11}}(K+k,k)^{-1}), \quad (A_{22}^{(k)}, A_{22}^{(k)} \Phi_{E_{22}^{-1}A_{22}}(K+k,k)^{-1})$$

have no eigenvalues in common and it is easy to see that this corresponds to

$$\lambda(\Phi_{E_{11}^{-1}A_{11}}(K+k,k)) \cap \lambda(\Phi_{E_{22}^{-1}A_{22}}(K+k,k)) = \emptyset.$$

Since the characteristic multipliers are independent of k, it suffices to require this condition to be satisfied for $k = 0$, i.e.,

$$\lambda(\Pi_{11}) \cap \lambda(\Pi_{22}) = \emptyset \tag{4.3}$$

with $\Pi_{ii} = \Phi_{E_{ii}^{-1}A_{ii}}(K, 0)$ as in (3.5). Also if some $E_{ii}^{(k)}$ happen to be singular, a continuity argument can be used to show that (4.3) is sufficient and necessary for the unique solvability of (3.7). Throughout this paper we assume that the condition (4) is fulfilled. If the condition is violated then, since (A_k, E_k) is in GPRSF, the eigenvalues of Π_{11} and Π_{22} are actually equal and there is in principle no need for swapping.

The condition (4) is equivalent to

$$\text{sep[PGCSY]} = \sigma_{\min}(Z_{\text{PGCSY}}) \neq 0. \tag{4.4}$$

As for deflating subspaces of regular matrix pairs (see, e.g., [32, 22]), the quantity sep[PGCSY] measures the sensitivity of the periodic deflating subspace pair of the GPRSF [5, 25, 33]. If K, p_1 or p_2 become large this quantity is very expensive to compute explicitly. By using the well-known estimation technique described in [16, 18, 21, 22], reliable sep[PGCSY]-estimates can be computed at the cost of solving a few PGCSYs.

5 Error analysis of K-cyclic equivalence swapping of diagonal blocks

In this section, we present an error analysis of the direct method described in Section 3 by extending the results in [20] to the case of periodic matrix pairs. We sometimes omit the index range $k = 0, 1, \ldots, K-1$, assuming that it is implicitly understood.

In finite precision arithmetic, the transformed matrix pair sequence will be affected by roundoff errors, resulting in a computed sequence $(\tilde{A}_k, \tilde{E}_k)$. We express the computed transformed matrix pairs as

$$(\tilde{A}_k, \tilde{E}_k) = (\hat{A}_k + \Delta A_k, \hat{E}_k + \Delta E_k),$$

where $(\hat{A}_k, \hat{E}_k)$ for $k = 0, \ldots K-1$ correspond to the exact matrix pairs in the reordered GPRSF of (A_k, B_k). Our task is to derive explicit expressions and upper bounds for the error matrices ΔA_k and ΔE_k. Most critical are of course the subdiagonal blocks of a 2×2 block partioned sequence $(\Delta A_k, \Delta E_k)$. These must be negligible in order to guarantee numerical backward stability for the swapping of diagonal blocks.

Let $(\tilde{L}_k, \tilde{R}_k) = (L_k + \Delta L_k, R_k + \Delta R_k)$ denote the computed solution to the associated PGCSY. The residual pair sequence of the computed solution is then given by $(Y_1^{(k)}, Y_2^{(k)})$, where

$$\left\{ \begin{array}{lcrcrcl} Y_1^{(k)} & \equiv & A_{11}^{(k)}\tilde{R}_k & - & \tilde{L}_k A_{22}^{(k)} & + & A_{12}^{(k)}, \\ Y_2^{(k)} & \equiv & E_{11}^{(k)}\tilde{R}_{k\oplus 1} & - & \tilde{L}_k E_{22}^{(k)} & + & E_{12}^{(k)}. \end{array} \right. \tag{5.1}$$

In addition, let $\tilde{Q}_k,\ \tilde{T}_L^{(k)}$ denote the computed factors of the kth QR factorization

$$\tilde{G}_L^{(k)} \equiv \begin{bmatrix} \tilde{L}_k \\ I_{p_2} \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} \tilde{T}_L^{(k)} \\ 0 \end{bmatrix}, \tag{5.2}$$

where $\tilde{Q}_k = Q_k + \Delta Q_k$, $\tilde{T}_L^{(k)} = T_L^{(k)} + \Delta T_L^{(k)}$ and $Q_k,\ T_L^{(k)}$ are the exact factors. Similarly, let $\tilde{Z}_k,\ \tilde{T}_R^{(k)}$ denote the computed factors of the kth RQ factorization

$$\tilde{G}_R^{(k)} \equiv \begin{bmatrix} I_{p_1} & -\tilde{R}_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{T}_R^{(k)} \end{bmatrix} \tilde{Z}_k^T, \tag{5.3}$$

where $\tilde{Z}_k = Z_k + \Delta Z_k$, $\tilde{T}_R^{(k)} = T_R^{(k)} + \Delta T_R^{(k)}$ and $Z_k,\ T_R^{(k)}$ are the exact factors. If Householder transformations are used to compute the factorizations (5.2)-(5.3), $\tilde{Q}_k$ and $\tilde{Z}_k$ are orthogonal to machine precision [40]. The error matrices ΔQ_k and ΔZ_k are essentially bounded by the condition numbers of $\tilde{G}_L^{(k)}$ and $\tilde{G}_R^{(k)}$, respectively, times the relative errors in these matrices (e.g., see [32, 19]).

We transform (A_k, E_k) using the computed $(\tilde{Q}_k, \tilde{Z}_k)$ in a K-cyclic equivalence transformation giving

$$\tilde{Q}^T(A_k, E_k)\tilde{Z}_k \quad = \quad (\hat{A}_k + \Delta A_k, \hat{E}_k + \Delta E_k), \tag{5.4}$$

where $(\hat{A}_k, \hat{E}_k)$ is the exact reordered GPRSF of the periodic (A_k, B_k) sequence. Our aim is to derive explicit expressions and norm bounds for blocks of $(\Delta A_k, \Delta E_k)$. First,

$$\begin{aligned} \tilde{Q}^T A_k \tilde{Z}_k &= (Q_k + \Delta Q_k)^T A_k (Z_k + \Delta Z_k) = \\ &= Q_k^T A_k Z_k + \Delta Q_k^T A_k Z_k + Q_k^T A_k \Delta Z_k + \Delta Q_k^T A_k \Delta Z_k, \end{aligned} \tag{5.5}$$

and by dropping the second order term and using $\hat{A}_k = Q_k^T A_k Z_k$ and $\Delta Q_k^T Q_k = -Q_k \Delta Q_k^T$ up to first order we get

$$\begin{aligned} &\tilde{Q}^T A_k \tilde{Z}_k = \hat{A}_k + \hat{A}_k(Z_k^T \Delta Z_k) + (-Q_k \Delta Q_k^T)\hat{A}_k = \hat{A}_k + \Delta A_k, \\ &\text{with } \Delta A_k \equiv \hat{A}_k U_k + W_k \hat{A}_k, \text{ where } U_k = Z_k^T \Delta Z_k \text{ and } W_k = -Q_k \Delta Q_k^T. \end{aligned} \tag{5.6}$$

Similarly, we get

$$\tilde{Q}^T B_k \tilde{Z}_{k\oplus 1} \quad = \quad \hat{E}_k + \Delta E_k \text{ with } \Delta E_k \equiv \hat{E}_k U_{k\oplus 1} + W_k \hat{E}_k. \tag{5.7}$$

After partitioning $U_k,\ U_{k\oplus 1},\ W_k$ and $(\Delta A_k, \Delta E_k)$ in conformity with $(\hat{A}_k, \hat{E}_k)$ and doing straightforward block matrix multiplications we get

$$\begin{aligned} \Delta A_{11}^{(k)} &= \hat{A}_{11} U_{11}^{(k)} + W_{11}^{(k)} \hat{A}_{11}^{(k)} + \hat{A}_{12}^{(k)} U_{21}^{(k)}, \\ \Delta A_{12}^{(k)} &= \hat{A}_{11}^{(k)} U_{12}^{(k)} + \hat{A}_{12}^{(k)} U_{22}^{(k)} + W_{11}^{(k)} \hat{A}_{12}^{(k)} + W_{12}^{(k)} \hat{A}_{22}^{(k)}, \\ \Delta A_{21}^{(k)} &= \hat{A}_{22}^{(k)} U_{21}^{(k)} + W_{21}^{(k)} \hat{A}_{11}^{(k)}, \\ \Delta A_{22}^{(k)} &= \hat{A}_{22}^{(k)} U_{22}^{(k)} + W_{22}^{(k)} \hat{A}_{22}^{(k)} + W_{21}^{(k)} \hat{A}_{12}^{(k)}, \end{aligned}$$

and

$$\begin{aligned}
\Delta E_{11}^{(k)} &= \hat{E}_{11} U_{11}^{(k\oplus 1)} + W_{11}^{(k)} \hat{E}_{11}^{(k)} + \hat{E}_{12}^{(k)} U_{21}^{(k\oplus 1)}, \\
\Delta E_{12}^{(k)} &= \hat{E}_{11}^{(k)} U_{12}^{(k\oplus 1)} + \hat{E}_{12}^{(k)} U_{22}^{(k\oplus 1)} + W_{11}^{(k)} \hat{E}_{12}^{(k)} + W_{12}^{(k)} \hat{E}_{22}^{(k)}, \\
\Delta E_{21}^{(k)} &= \hat{E}_{22}^{(k)} U_{21}^{(k\oplus 1)} + W_{21}^{(k)} \hat{E}_{11}^{(k)}, \\
\Delta E_{22}^{(k)} &= \hat{E}_{22}^{(k)} U_{22}^{(k\oplus 1)} + W_{22}^{(k)} \hat{E}_{22}^{(k)} + W_{21}^{(k)} \hat{E}_{12}^{(k)}.
\end{aligned}$$

Observe that $\Delta A_{11}^{(k)}$, $\Delta A_{22}^{(k)}$, $\Delta E_{11}^{(k)}$, $\Delta E_{22}^{(k)}$ affect the reordered K-cyclic diagonal block pairs and possibly the eigenvalues, while $\Delta A_{21}^{(k)}$ and $\Delta E_{21}^{(k)}$ are even more critical since they affect the eigenvalues as well as the stability of the reordering; these are the perturbations of interest that we investigate further. The analysis in [20] applied to (5.2)–(5.3), results in the following expressions for blocks of U_k and W_k:

$$\begin{aligned}
U_{11}^{(k)} &= -{Z_{21}^{(k)}}^{-1} Z_{22}^{(k)} {T_R^{(k)}}^{-1} \Delta R_k Z_{21}^{(k)}, \\
U_{21}^{(k)} &= {T_R^{(k)}}^{-1} \Delta R_k Z_{21}^{(k)}, \\
U_{22}^{(k)} &= {T_R^{(k)}}^{-1} \Delta R_k Z_{22}^{(k)},
\end{aligned}$$

and

$$\begin{aligned}
W_{11}^{(k)} &= -{Q_{11}^{(k)}}^{T} \Delta L_k {T_L^{(k)}}^{-1}, \\
W_{21}^{(k)} &= -{Q_{12}^{(k)}}^{T} \Delta L_k {T_L^{(k)}}^{-1}, \\
W_{22}^{(k)} &= {Q_{12}^{(k)}}^{T} {T_L^{(k)}}^{-1} {Q_{11}^{(k)}}^{T} {Q_{12}^{(k)}}^{-T},
\end{aligned}$$

up to first order perturbations. By substituting the expressions for $U_{ij}^{(k)}$ and $W_{ij}^{(k)}$ in $\Delta A_{ij}^{(k)}, \Delta E_{ij}^{(k)}$ we obtain

$$\Delta A_{11}^{(k)} = {Q_{11}^{(k)}}^{T} Y_1^{(k)} Z_{21}^{(k)}, \tag{5.8}$$

$$\Delta A_{21}^{(k)} = {Q_{12}^{(k)}}^{T} Y_1^{(k)} Z_{21}^{(k)}, \tag{5.9}$$

$$\Delta A_{22}^{(k)} = {Q_{12}^{(k)}}^{T} Y_1^{(k)} Z_{22}^{(k)}, \tag{5.10}$$

and

$$\Delta E_{11}^{(k)} = {Q_{11}^{(k)}}^{T} Y_2^{(k)} Z_{21}^{(k\oplus 1)}, \tag{5.11}$$

$$\Delta E_{21}^{(k)} = {Q_{12}^{(k)}}^{T} Y_2^{(k)} Z_{21}^{(k\oplus 1)}, \tag{5.12}$$

$$\Delta E_{22}^{(k)} = {Q_{12}^{(k)}}^{T} Y_2^{(k)} Z_{22}^{(k\oplus 1)}, \tag{5.13}$$

with the residuals $(Y_1^{(k)}, Y_2^{(k)})$ as in (5.1). From the QR and RQ factorizations (3.12) we have

$$Q_{21}^{(k)} = {T_L^{(k)}}^{-1}, \quad {T_L^{(k)}}^{T} T_L^{(k)} = I_{p_2} + L_k^T L_k, \tag{5.14}$$

and

$$Z_{12}^{(k)^T} = T_R^{(k)^{-1}}, \quad T_R^{(k)} T_R^{(k)^T} = I_{p_1} + R_k R_k^T. \tag{5.15}$$

From (5.14)–(5.15) we obtain the following relations between the singular values of $T_L^{(k)}, T_R^{(k)}, L_k$ and R_k:

$$\sigma^2(T_L^{(k)}) = 1 + \sigma^2(L_k), \quad \sigma^2(T_R^{(k)}) = 1 + \sigma^2(R_k). \tag{5.16}$$

Further, from the CS decomposition (see, e.g., [12]) of Q_k and Z_k, respectively, we obtain the relations

$$\|Q_{12}^{(k)^T}\|_2 = \|Q_{21}^{(k)}\|_2, \quad \|Q_{22}^{(k)}\|_2 = \|Q_{11}^{(k)}\|_2,$$
$$\|Z_{12}^{(k)^T}\|_2 = \|Z_{21}^{(k)}\|_2, \quad \|Z_{22}^{(k)}\|_2 = \|Z_{11}^{(k)}\|_2.$$

Combining these results, we get

$$\|Q_{12}^{(k)^T}\|_2 = \|T_L^{(k)^{-1}}\|_2 = \frac{1}{\sigma_{\min}(T_L^{(k)})} = \frac{1}{(1+\sigma_{\min}^2(L_k))^{1/2}},$$
$$\|Q_{11}^{(k)}\|_2 = \frac{\sigma_{\max}(L_k)}{(1+\sigma_{\max}^2(L_k))^{1/2}},$$

and

$$\|Z_{21}^{(k)}\|_2 = \|T_R^{(k)^{-1}}\|_2 = \frac{1}{\sigma_{\min}(T_R^{(k)})} = \frac{1}{(1+\sigma_{\min}^2(R_k))^{1/2}},$$
$$\|Z_{22}^{(k)}\|_2 = \frac{\sigma_{\max}(R_k)}{(1+\sigma_{\max}^2(R_k))^{1/2}},$$

and we have proved the following theorem by applying the submultiplicativity of matrix norms to (5.8)–(5.13).

THEOREM 5.1. *After applying the computed transformation matrices $\tilde{Q}_k, \tilde{Z}_k$ from (5.2)-(5.3) in a K-cyclic equivalence transformation of (A_k, E_k) defined in (3.1), we get*

$$\tilde{Q}_k^T A_k \tilde{Z}_k = \tilde{A}_k, \text{ where } \tilde{A}_k \equiv \hat{A}_k + \Delta A_k = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta A_{11}^{(k)} & \Delta A_{12}^{(k)} \\ \Delta A_{21}^{(k)} & \Delta A_{22}^{(k)} \end{bmatrix},$$
$$\tilde{Q}_k^T E_k \tilde{Z}_{k\oplus 1} = \tilde{E}_k, \text{ where } \tilde{E}_k \equiv \hat{E}_k + \Delta E_k = \begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta E_{11}^{(k)} & \Delta E_{12}^{(k)} \\ \Delta E_{21}^{(k)} & \Delta E_{22}^{(k)} \end{bmatrix}.$$

The critical blocks of the error matrix pair $(\Delta A_k, \Delta E_k)$ satisfy the following error bounds, up to first order perturbations:

$$\|\Delta A_{11}^{(k)}\|_2 \le \frac{\sigma_{\max}(L_k)}{(1+\sigma_{\max}^2(L_k))^{1/2}} \cdot \frac{1}{(1+\sigma_{\min}^2(R_k))^{1/2}} \cdot \|Y_1^{(k)}\|_F,$$
$$\|\Delta A_{21}^{(k)}\|_2 \le \frac{1}{(1+\sigma_{\min}^2(L_k))^{1/2}} \cdot \frac{1}{(1+\sigma_{\min}^2(R_k))^{1/2}} \cdot \|Y_1^{(k)}\|_F,$$
$$\|\Delta A_{22}^{(k)}\|_2 \le \frac{1}{(1+\sigma_{\min}^2(L_k))^{1/2}} \cdot \frac{\sigma_{\max}(R_k)}{(1+\sigma_{\max}^2(R_k))^{1/2}} \cdot \|Y_1^{(k)}\|_F,$$

and

$$\|\Delta E_{11}^{(k)}\|_2 \leq \frac{\sigma_{\max}(L_k)}{(1+\sigma_{\max}^2(L_k))^{1/2}} \cdot \frac{1}{(1+\sigma_{\min}^2(R_{k\oplus 1}))^{1/2}} \cdot \|Y_2^{(k)}\|_F,$$
$$\|\Delta E_{21}^{(k)}\|_2 \leq \frac{1}{(1+\sigma_{\min}^2(L_k))^{1/2}} \cdot \frac{1}{(1+\sigma_{\min}^2(R_{k\oplus 1}))^{1/2}} \cdot \|Y_2^{(k)}\|_F,$$
$$\|\Delta E_{22}^{(k)}\|_2 \leq \frac{1}{(1+\sigma_{\min}^2(L_k))^{1/2}} \cdot \frac{\sigma_{\max}(R_{k\oplus 1})}{(1+\sigma_{\max}^2(R_{k\oplus 1}))^{1/2}} \cdot \|Y_2^{(k)}\|_F,$$

for $k = 0, 1, \ldots, K-1$. *Moreover, the matrix pair sequences* $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)}), (A_{22}^{(k)}, E_{22}^{(k)})$ *and* $(A_{11}^{(k)}, E_{11}^{(k)}), (\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ *are* K*-cyclic equivalent and have the same generalized eigenvalues, respectively.*

REMARK 5.1. *Theorem 5.1 shows that the stability and accuracy of the reordering method is governed mainly by the conditioning and accuracy of the solution to the associated PGCSY. The errors* $\|\Delta A_{ij}^{(k)}\|_2$ *and* $\|\Delta E_{ij}^{(k)}\|_2$ *can be as large as the norm of the residuals* $\|Y_1^{(k)}\|_F$ *and* $\|Y_2^{(k)}\|_F$*, respectively. Indeed, this happens when the smallest singular values of the exact sequences* L_k *and* R_k *are tiny, indicating an ill-conditioned underlying PGCSY equation. We have experimental evidence that* $\|Y_1^{(k)}\|_F$ *and* $\|Y_2^{(k)}\|_F$ *can be large for large-normed (ill-conditioned) solutions of the associated PGCSY. In the next section, we show how we handle such situations and guarantee backward stability of the periodic reordering method.*

REMARK 5.2. *For periodicity* $K = 1$*, Theorem 5.1 reduces to the main theorem of [20] on the perturbation of the generalized eigenvalues under eigenvalue reordering in the generalized real Schur form of a regular matrix pencil.*

6 Algorithms and implementation aspects

In this section, we address some implementation issues of the direct method for reordering eigenvalues in a generalized periodic real Schur form described and analyzed in the previous sections.

6.1 Algorithms for solving the PGCSY

The linear system (4.1) that arises from the PGCSY (3.7) has a pariticular structure that needs to be exploited in order to keep the cost of the overall algorithm linear in K. The matrix Z_{PGCSY} in (4.1) belongs to the class of *bordered almost block diagonal* (BABD) matrices, which takes the more general

form

$$(6.1)\quad Z = \begin{bmatrix} Z_{0,0} & & & & & & Z_{0,2K-1} \\ Z_{1,0} & Z_{1,1} & & & & & \\ & Z_{2,1} & Z_{2,2} & & & & \\ & & Z_{3,2} & Z_{3,3} & & & \\ & & & Z_{4,3} & \ddots & & \\ & & & & \ddots & & \\ & & & & & Z_{2K-2,2K-2} & \\ & & & & & Z_{2K-1,2K-2} & Z_{2K-1,2K-1} \end{bmatrix},$$

where each nonzero block $Z_{i,j}$ is $m \times m$ (note that $m = p_1 p_2$ for the matrix Z_{PGCSY} of size $2Km \times 2Km$). An overview of numerical methods that address linear systems with such BABD structure is given in [10]. Gaussian elimination with partial pivoting, for example, preserves much of the structure of Z and can be implemented very efficiently. Unfortunately, matrices with BABD structure happen to be among the rare examples of practical relevance which may lead to numerical instabilities because of excessive pivot growth [42]. Gaussian elimination with complete pivoting avoids this phenomenom but is too expensive both in terms of cost and storage space. In contrast, structured variants of the QR factorization are both numerically stable and efficient [11, 41]. In the following, we describe such a structured QR factorization in more detail.

Algorithm 6.1 Overlapping QR factorization of the BABD-system $Zx = y$

Input:	Matrix $Z \in \mathbb{R}^{2Km \times 2Km}$, right hand side vector $y \in \mathbb{R}^{2Km}$.
Output:	Orthogonal transformations $\bar{Q}_k \in \mathbb{R}^{2m \times 2m}$, $k = 0, 1, \ldots, 2K-2$, $\bar{Q}_{2K-1} \in \mathbb{R}^{m \times m}$, triangular factor $\bar{R} \in \mathbb{R}^{2Km \times 2Km}$ with structure as in Equation (6.2), vector $\bar{y} \in \mathbb{R}^{2Km}$ such that $\bar{R}x = \bar{y}$.

for $k = 0$ up to $2K-2$ **do**
  QR factorize: $\bar{Q}_k \bar{R}_k = \begin{bmatrix} Z_{k,k}^T, & Z_{k+1,k}^T \end{bmatrix}^T$
  Update: $\begin{bmatrix} Z_{k,k+1}^T, & Z_{k+1,k+1}^T \end{bmatrix}^T = \bar{Q}_k^T \begin{bmatrix} Z_{k,k+1}^T, & Z_{k+1,k+1}^T \end{bmatrix}^T$
  Update: $\begin{bmatrix} Z_{k,K-1}^T, & Z_{k+1,K-1}^T \end{bmatrix}^T = \bar{Q}_k^T \begin{bmatrix} Z_{k,K-1}^T, & Z_{k+1,K-1}^T \end{bmatrix}^T$
  Update right hand side: $\bar{y}_k = \bar{Q}_k^T y_k$
end for
QR factorize: $\bar{Q}_{2K-1} \bar{R}_{2K-1} = Z_{2K-1,2K-1}$
Update right hand side: $\bar{y}_{2K-1} = \bar{Q}_{2K-1}^T y_{2K-1}$

To solve a linear system $Zx = y$, we first reduce the matrix Z in (6.1) to upper triangular form. For this purpose, we successively apply Householder transformations to reduce each block $\begin{bmatrix} Z_{k,k}^T, & Z_{k+1,k}^T \end{bmatrix}^T$, $k = 0, 1, \ldots, K-2$, to upper trapezoidal form, and the block $Z_{2K-1,2K-1}$ to upper triangular form. Each computed Householder transformation is applied to the corresponding block row (as well as the right hand side y of the equation, which is blocked in conformity with Z) before the next transformation is computed. The factorization procedure is outlined in Algorithm 6.1, where for simplicity of presentation the

Householder transformations are accumulated into orthogonal transformation matrices $\bar{Q}_k$.

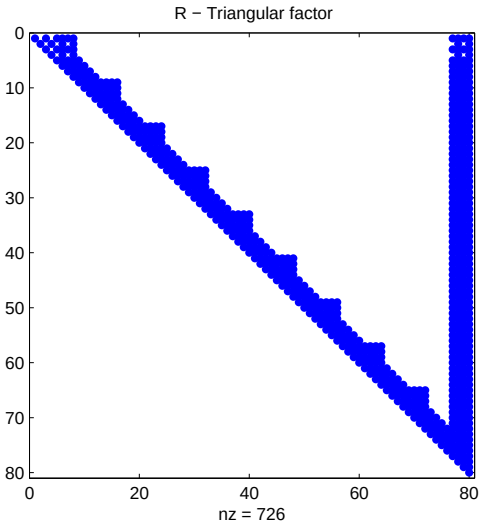

Figure 6.1: The resulting R-factor from applying overlapping QR factorizations to the matrix Z_{PGCSY} for $K = 10$, $p_1 = p_2 = 2$, visualized by the Matlab `spy` command. The "sawtooth" above the main block diagonal is typical for the PGCSY and does not occur in the case of periodic matrix reordering [14].

Algorithm 6.2 Backward substitution for solving $\bar{R}x = \bar{y}$

Input: Matrix $\bar{R} \in \mathbb{R}^{2Km \times 2Km}$, with the structure displayed in Equation (6.2), right hand side vector $\bar{y} \in \mathbb{R}^{2Km}$ partitioned in conformity with the structure of $\bar{R}$.
Output: Solution vector $x \in \mathbb{R}^{2Km}$ such that $\bar{R}x = \bar{y}$.

for $k = K - 1$ down to 0 **do**
  Solve: $x_{2k+1} = [\bar{R}_{2k+1}]^{-1}\bar{y}_{2k+1}$ using backward substitution ($\bar{R}_{2k+1} \leftarrow I_{m\times m}$)
  Update: $\bar{y}_{2k} = \bar{y}_{2k} - G_k x_{2k+1}$ ($G_k \leftarrow 0$)
  if $(k = K - 1)$ **then**
    for $i = K - 2$ down to 0 **do**
      Update: $\bar{y}_{2k+1} = \bar{y}_{2k+1} - F_{2k+1}x_{2K-1}$ ($F_{2k+1} \leftarrow 0$)
      Update: $\bar{y}_{2k} = \bar{y}_{2k} - F_{2k}x_{2K-1}$ ($F_{2k} \leftarrow 0$)
    end for
  end if
  Solve: $x_{2k} = [\bar{R}_{2k}]^{-1}\bar{y}_{2k}$ using backward substitution ($\bar{R}_{2k} \leftarrow I_{m\times m}$)
  if $(k > 0)$ **then**
    Update: $\bar{y}_{2k-1} = \bar{y}_{2k-1} - \bar{L}_{k-1}x_{2k}$ ($\bar{L}_{k-1} \leftarrow 0$)
  end if
end for

It is straightforward to see that this procedure of computing overlapping orthogonal factorizations produces the same amount of fill-in elements in the rightmost block columns of Z as would GEPP produce in the worst case, see also Figure 6.1. More formally, the QR factorization reduces the matrix Z into the following form:

$$
\begin{bmatrix}
\bar{R}_0 & G_0 & & & & & & & & F_0 \\
 & \bar{R}_1 & \bar{L}_0 & & & & & & & F_1 \\
 & & \bar{R}_2 & G_1 & & & & & & F_2 \\
 & & & \bar{R}_3 & \bar{L}_1 & & & & & F_3 \\
 & & & & & \ddots & & & & \vdots \\
 & & & & & \ddots & & & & \vdots \\
 & & & & & & \bar{R}_{2K-4} & G_{K-2} & & F_{2K-4} \\
 & & & & & & & \bar{R}_{2K-3} & \bar{L}_{K-2} & F_{2K-3} \\
 & & & & & & & & \bar{R}_{2K-2} & G_{K-1} \\
 & & & & & & & & & \bar{R}_{2K-1}
\end{bmatrix}, \tag{6.2}
$$

with $\bar{R}_k, \bar{L}_k, F_k, G_k \in \mathbb{R}^{m\times m}$: $\bar{R}_k$ $(k = 0, 1, \ldots, 2K-1)$ are upper triangular, whereas $\bar{L}_k$ $(k = 0, 1, \ldots, K-2)$, G_k, $(k = 0, 1, \ldots, K-1)$, and F_k $(k = 0, 1, \ldots, 2K-3)$ are dense matrices. Moreover, the blocks $\bar{L}_k$ are lower triangular provided that $Z_{2,2}, Z_{4,4}, \ldots, Z_{2K-2,2K-2}$ and $Z_{2,1}, Z_{4,3}, \ldots, Z_{2K-2,2K-1}$ in (6.1) are lower and upper triangular, respectively, which is the case for Z_{PGCSY}. To compute x we employ backward substitution on this structure, as outlined in Algorithm 6.2.

We remark that the new algorithms described here for solving small-sized PGCSY equations can be used as kernel solvers in recursive blocked algorithms [13] for solving large-scale problems.

REMARK 6.1. *Solving a linear system with QR factorization yields a small norm-wise backward error [19], i.e., the computed solution $\hat{x}$ is the exact solution of a slightly perturbed system $(Z + \triangle Z)\hat{x} = y$, where $\|\triangle Z\|_F = O(\mathbf{u}\|Z\|_F)$ with $\mathbf{u}$ denoting the unit roundoff. However, the standard implementation of the QR factorization is not row-wise backward stable, i.e., the norm of a row in $\triangle Z$ may not be negligible compared to the norm of the corresponding row in Z. This may cause instabilities if the norms of the coefficient matrices A_k, E_k differ significantly. To avoid this effect, we scale each A_k and E_k to Frobenius norm 1 before solving (3.7). Then each block row in Z_{PGCSY} has Frobenius norm at most $\sqrt{2}$ amd $\|Z_{\mathrm{PGCSY}}\|_F \leq 2\sqrt{K}$. The resulting swapping transformation is applied to the original unscaled K-cyclic matrix pair sequence. The corresponding residuals satisfy*

$$\|Y_1^{(k)}\|_F = O(\mathbf{u}\|A_k\|_F\|(L_k, R_k)\|_F), \quad \|Y_2^{(k)}\|_F = O(\mathbf{u}\|E_k\|_F\|(L_k, R_{k\oplus 1})\|_F).$$

Combined with Theorem 5.1, this shows that the backward error of the developed reordering method is norm-wise small for each coefficient A_k and E_k, unless (3.7) is too ill-conditioned.

6.2 K-cyclic equivalence swapping algorithm with stability tests

Considering the error analysis in Section 5 and in the spirit of [22, 14], we formulate stability test criteria for deciding whether a K-cyclic equivalence swap should be accepted or not.

From Equation (3.12) and the following partition of the transformation matrix sequences Q_k and Z_k, we obtain the relations

$$L_k Q_{21}^{(k)} - Q_{11}^{(k)} = 0, \quad {Z_{12}^{(k)}}^T R_k + {Z_{22}^{(k)}}^T = 0, \tag{6.3}$$

which can be computed before the swapping is performed. We use computed quantities of these relations to define the *weak stability criterion*:

$$R_{\mathrm{weak}} = \max_{0 \le k \le K-1} \max \left(\frac{\|\tilde{L}_k \tilde{Q}_{21}^{(k)} - \tilde{Q}_{11}^{(k)}\|_F}{\|\tilde{L}_k\|_F}, \frac{\|{\tilde{Z}_{12}^{(k)}}^T \tilde{R}_k + {\tilde{Z}_{22}^{(k)}}^T\|_F}{\|\tilde{R}_k\|_F} \right). \tag{6.4}$$

We remark that the relative criterion R_{weak} should be small even for ill-conditioned PGSCY equations with large normed solutions L_k and R_k (see also Remarks 5.1 and 6.1). After the swap has been performed, the maximum residual over the whole K-period defines a *strong stability criterion*:

$$R_{\mathrm{strong}} = \max_{0 \le k \le K-1} \max \left(\frac{\|A_k - \tilde{Q}_k \tilde{A}_k \tilde{Z}_k^T\|_F}{\|A_k\|_F}, \frac{\|E_k - \tilde{Q}_k \tilde{E}_k \tilde{Z}_{k\oplus 1}^T\|_F}{\|E_k\|_F} \right). \tag{6.5}$$

If both R_{weak} and R_{strong} are less than a specified tolerance ε_{u} (a small constant times the machine precision), the swap is accepted, otherwise it is rejected. In this way, backward stability is guaranteed for the K-cyclic equivalence swapping.

In summary, we have the following algorithm for swapping two matrix pair sequences of diagonal blocks in the GPRSF of a regular K-cyclic matrix pair (A_k, B_k) of size $(p_1 + p_2) \times (p_1 + p_2)$:

1. *Compute K-cyclic matrix pair sequence $(\tilde{L}_k, \tilde{R}_k)$ by solving the scaled PGCSY (3.7) using Algorithm 6.1 and Algorithm 6.2.*

2. *Compute K-cyclic orthogonal matrix sequence $\tilde{Q}_k$ using QR factorizations:*

$$\begin{bmatrix} \tilde{L}_k \\ I_{p_2} \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} \tilde{T}_L^{(k)} \\ 0 \end{bmatrix}, \ k = 0, 1, \ldots, K-1.$$

3. *Compute K-cyclic orthogonal matrix sequence $\tilde{Z}_k$ using RQ factorizations:*

$$\begin{bmatrix} I_{p_1} & -\tilde{R}_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{T}_R^{(k)} \end{bmatrix} \tilde{Z}_k^T, \ k = 0, 1, \ldots, K-1.$$

4. *Compute* $(\tilde{A}, \tilde{E}) = (\tilde{Q}_k^T A_k \tilde{Z}_k, \tilde{Q}_k^T E_k \tilde{Z}_{k\oplus 1})$ *for* $k = 0, 1, \ldots, K-1$, *i.e., an orthogonal* K*-cyclic equivalence transformation of* (A_k, E_k)*:*

$$\tilde{A} \equiv \begin{bmatrix} \tilde{A}_{11}^{(k)} & \tilde{A}_{12}^{(k)} \\ \tilde{A}_{21}^{(k)} & \tilde{A}_{22}^{(k)} \end{bmatrix} = \tilde{Q}_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} \tilde{Z}_k,$$

$$\tilde{E} \equiv \begin{bmatrix} \tilde{E}_{11}^{(k)} & \tilde{E}_{12}^{(k)} \\ \tilde{E}_{21}^{(k)} & \tilde{E}_{22}^{(k)} \end{bmatrix} = \tilde{Q}_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} \tilde{Z}_{k\oplus 1}.$$

5. *If* $R_{\text{weak}} < \varepsilon_{\text{u}} \wedge R_{\text{strong}} < \varepsilon_{\text{u}}$, accept swap *and*

5a. *set* $\tilde{A}_{21}^{(k)} = \tilde{E}_{21}^{(k)} = 0$,

5b. *restore GPRSF of* $(\tilde{A}_{11}^{(k)}, \tilde{E}_{11}^{(k)})$ *and* $(\tilde{A}_{22}^{(k)}, \tilde{E}_{22}^{(k)})$ *by the periodic QZ algorithm;*

otherwise reject swap.

The stability tests in step 5 for accepting a K-cyclic swap guarantee that the subdiagonal blocks $\tilde{A}_{21}^{(k)}$ and $\tilde{E}_{21}^{(k)}$ are negligible compared to the rest of the matrices. Step 5b can be performed by a fixed number of operations for adjacent diagonal blocks in the GPRSF, i.e., for $p_i \in \{1, 2\}$ (see [14] for the standard periodic matrix case).

Properly implemented, this algorithm requires $O(K)$ floating point operations (flops), where K is the periodicity. When it is used to reorder two adjacent diagonal blocks in a larger $n \times n$ periodic matrix pair in GPRSF then the off-diagonal parts are updated by the transformation matrices $\tilde{Q}_k$ and $\tilde{Z}_k$, which additionally requires $O(Kn)$ flops.

There are several other important implementation issues to be considered for a completely reliable implementation. For example, iterative refinement in extended precision arithmetic can be used to improve the accuracy of the PGCSY solution and avoid the possibility of rejection (see, e.g., [19]). Our experiences so far concern iterative refinement in standard precision arithmetic and (as expected) the results show no substantial improvements.

7 Computational experiments

The direct reordering algorithm described in the previous sections has been implemented in MATLAB. A more robust and efficient Fortran implementation will be included in a forthcoming software toolbox for periodic eigenvalue problems. In this section, we present some numerical results using our prototype implementation. All experiments were carried out in double precision ($\epsilon_{\text{mach}} \approx 2.2 \times 10^{-16}$).

The test examples range from well-conditioned to ill-conditioned problems, including matrix pair sequences of small and large periodicity. In Table 7.1,

we display some problem characteristics: problem dimension n (2, 3 or 4 corresponding to swapping a mix of 1×1 and 2×2 blocks), periodicity K, the computed value of sep[PGCSY] $= \sigma_{\min}(Z_{\mathrm{PGCSY}})$ (see Section 4) and

$$s = 1/\sqrt{1 + \|(L_0, R_0)\|_F^2},$$

where (L_0, R_0) is the generator matrix pair (see Section 3.2). The quantities s and sep[PGCSY] partly govern the sensitivity of the selected eigenvalues and associated periodic deflating subspaces, see [5, 25, 33].

The results from the periodic reordering are presented in Table 7.2. These include the weak (R_{weak}) and strong (R_{strong}) stability tests, the residual norms for the GPRSF before (R_{gprsf}) and after (R_{reord}) the reordering computed as in Equation (6.5), a relative orthogonality check of the accumulated transformations after (R_{orth}) the reordering computed as

$$R_{\mathrm{orth}} = \frac{\max_k(\|I_{n_k} - \tilde{W}_k^T \tilde{W}_k\|_F, \|I_{n_k} - \tilde{W}_k \tilde{W}_k^T\|_F)}{\epsilon_{\mathrm{mach}}},$$

where the maximum is taken over the period K for all transformation matrices $\tilde{Q}_k$ and $\tilde{Z}_k$. The last column displays the maximum relative change of the eigenvalues after the periodic reordering

$$R_{\mathrm{eig}} = \max_k \frac{|\lambda_k - \tilde{\lambda}_k|}{|\lambda_k|}, \ \lambda_k \in \lambda(\Phi_{E^{-1}A}(K, 0)).$$

Notice that we normally do not compute λ_i explicitly but keep it as an eigenvalue pair (α_i, β_i) to avoid losing information because of roundoff errors. This is especially important for tiny and large values of α_i and/or β_i.

The eigenvalues before and after reordering are shown in full precision under each example. For 2×2 matrix sequences, we compute the generalized eigenvalues via unitary transformations in the GPRSF as is done in LAPACK's `DTGSEN` [1].

Table 7.1: Problem characteristics.

Example	n	K	sep[PGCSY]	s
I	2	2	1.1E-8	1.4E-4
II	4	10	3.3E-2	4.9E-1
III	4	100	1.4E-3	1.9E-1
IV	4	100	1.4E-14	6.1E-7
V	3	5	7.1E-2	6.2E-1
VI	2	50	1.6E-2	5.8E-1

Example I. Consider the following sequence with $n = 2, K = 2$:

$$A_1 = \begin{bmatrix} 2\epsilon^{1/2} & -1 \\ 0 & -2\epsilon^{1/2} \end{bmatrix}, A_2 = E_1 = E_2 = \begin{bmatrix} \epsilon^{1/2} & 1 \\ 0 & \epsilon^{1/2} \end{bmatrix}.$$

Table 7.2: Reordering results using QR factorization to solve the associated PGCSY.

Example	R_{weak}	R_{strong}	R_{gprsf}	R_{reord}	R_{orth}	R_{eig}
I	6.3E-17	5.0E-16	0	5.0E-16	2.0	3.2E-9
II	1.6E-16	9.0E-16	4.8E-15	5.6E-15	7.5	4.6E-15
III	1.8E-16	1.3E-15	2.2E-16	3.2E-15	8.3	3.3E-14
IV	8.3E-17	1.0E-15	2.2E-16	2.4E-15	7.6	3.8E-14
V	1.3E-16	7.0E-16	8.3E-17	9.1E-16	2.8	1.8E-15
VI	3.8E-16	8.2E-16	0	9.8E-16	2.0	1.1E-16

This product has the (α, β)-pairs

$$\begin{aligned}(\alpha_1, \beta_1) &= (4.4408920985006, 2.2204460492503) \times 10^{-16},\\ (\alpha_2, \beta_2) &= (-4.4408920985006, -2.2204460492503) \times 10^{-16},\end{aligned}$$

which correspond to well-defined eigenvalues $\lambda_1 = 2.0$ and $\lambda_2 = -2.0$. But all α_i and β_i are at the machine precision level and this fact signals an obvious risk for losing accuracy after the reordering:

$$\begin{aligned}(\tilde{\alpha}_1, \tilde{\beta}_1) &= (9.5161972853921, -4.7580986273341) \times 10^{-16},\\ (\tilde{\alpha}_2, \tilde{\beta}_2) &= (-2.0724163126336, -1.0362081563168) \times 10^{-16},\end{aligned}$$

which define the eigenvalues $\tilde{\lambda}_1 = -2.00000000645717$ and $\tilde{\lambda}_2 = 2.00000000000000$.
Example II. Consider reordering the eigenvalues $\lambda_{1,2} = 2 \pm 2i$ and $\lambda_{3,4} = 1 \pm i$ in a matrix pair sequence with dimension $n = 4$ and period $K = 10$. The computed eigenvalues from the GPRSF are correct to full machine precision. After reordering we get the following (α, β)-pairs:

$$\begin{aligned}(\tilde{\alpha}_1, \tilde{\beta}_1) &= (-6.69743899940721 - 6.69743899940718i, -6.69743899940718),\\ (\tilde{\alpha}_2, \tilde{\beta}_2) &= (1.03550511685258 - 1.03550511685258i, 1.03550511685258),\\ (\tilde{\alpha}_3, \tilde{\beta}_3) &= (1.93142454580911 + 1.93142454580911i, 0.96571227290455),\\ (\tilde{\alpha}_4, \tilde{\beta}_4) &= (0.29862160747967 - 0.29862160747967i, 0.14931080373983).\end{aligned}$$

A quick check reveals that these pairs correspond to a reordering at almost full machine precision.
Example III. The eigenvalue pair $\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}i$ is located on the unit circle. In LQ-optimal control (see Section 2) we want to compute a periodic deflating subspace corresponding to the *stable* eigenvalues, i.e., the eigenvalues *inside* the unit disc.

For illustration, consider reordering the eigenvalues $\lambda_{1,2} = (\cos\frac{\pi}{4}+\delta) \pm (\sin\frac{\pi}{4}+\delta)i$ and $\lambda_{3,4} = (\cos\frac{\pi}{4}-\delta) \pm (\sin\frac{\pi}{4}-\delta)i$, where $\delta \in [0, 1]$, in a matrix pair sequence of period $K = 100$ arising, for example, from performing multi-rate sampling of a continuous-time system. At first, let $\delta = 10^{-1}$. The matrix product has the

computed (α, β)-pairs

$$\begin{aligned}
(\alpha_1, \beta_1) &= (0.80710678118654 + 0.80710678118654i, 1.00000000000002),\\
(\alpha_2, \beta_2) &= (0.80710678118654 - 0.80710678118654i, 1.00000000000002),\\
(\alpha_3, \beta_3) &= (-0.60710678118655 - 0.60710678118655i, -0.99999999999999),\\
(\alpha_4, \beta_4) &= (-0.60710678118655 + 0.60710678118655i, -1.00000000000000),
\end{aligned}$$

which correspond to the eigenvalues $\lambda_{1,2} = 0.80710678118652 \pm 0.80710678118652i$ and $\lambda_{3,4} = 0.60710678118655 \pm 0.60710678118655i$. After reordering we have

$$\begin{aligned}
(\tilde{\alpha}_1, \tilde{\beta}_1) &= (-1.53524924293502 - 1.53524924293503i, -2.52879607098851),\\
(\tilde{\alpha}_2, \tilde{\beta}_2) &= (-6.49961741950939 + 6.49961741950943i, -10.70588835592705),\\
(\tilde{\alpha}_3, \tilde{\beta}_3) &= (-0.07538905267396 - 0.07538905267396i, -0.09340654103182),\\
(\tilde{\alpha}_4, \tilde{\beta}_4) &= (0.31916641695471 - 0.31916641695471i, 0.39544509400044),
\end{aligned}$$

which define the eigenvalues $\tilde{\lambda}_{1,2} = 0.60710678118654 \pm 0.60710678118655i$ and $\tilde{\lambda}_{3,4} = 0.80710678118654 \pm 0.80710678118654i$.

Example IV. We consider Example III again, now with $\delta = 10^{-12}$. The matrix product has the computed (α, β)-pairs

$$\begin{aligned}
(\alpha_1, \beta_1) &= (-0.70710678118754 - 0.70710678118754i, -1.00000000000002),\\
(\alpha_2, \beta_2) &= (-0.70710678118755 + 0.70710678118755i, -0.99999999999999),\\
(\alpha_3, \beta_3) &= (0.70710678118555 + 0.70710678118555i, 1.00000000000000),\\
(\alpha_4, \beta_4) &= (-0.70710678118555 + 0.70710678118555i, -1.00000000000000),
\end{aligned}$$

which define the eigenvalues $\lambda_{1,2} = 0.70710678118755 \pm 0.70710678118754i$ and $\lambda_{3,4} = 0.70710678118555 \pm 0.70710678118555i$. After reordering we have

$$\begin{aligned}
(\tilde{\alpha}_1, \tilde{\beta}_1) &= (-0.70710678121274 - 0.70710678121274i, -1.00000000003845),\\
(\tilde{\alpha}_2, \tilde{\beta}_2) &= (0.70710678121274 - 0.70710678121274i, 1.00000000003845),\\
(\tilde{\alpha}_3, \tilde{\beta}_3) &= (-0.70710678116035 - 0.70710678116036i, -0.99999999996155),\\
(\tilde{\alpha}_4, \tilde{\beta}_4) &= (-0.70710678116036 + 0.70710678116036i, -0.99999999996155),
\end{aligned}$$

which correspond to the eigenvalues $\tilde{\lambda}_{1,2} = 0.70710678118555 \pm 0.70710678118555i$ and $\tilde{\lambda}_{3,4} = 0.70710678118754 \pm 0.70710678118755i$. The eigenvalues outside and inside the unit disc come closer and closer with a decreasing δ and the problem gets more ill-conditioned but we are still able to reorder the eigenvalues with satisfying accuracy. We illustrate the situation in Figure 7.

Example V. Consider reordering the following single eigenvalue $\lambda_1 = \sqrt{3}$ with the eigenvalue pair $\lambda_{2,3} = \frac{\sqrt{3}}{2} \pm \frac{1}{\sqrt{7}}i$. The original (α, β)-pairs are

$$\begin{aligned}
(\alpha_1, \beta_1) &= (1.73205080756888, 1.00000000000000),\\
(\alpha_2, \beta_2) &= (-0.86602540378444 - 0.37796447300923i, -1.00000000000000),\\
(\alpha_3, \beta_3) &= (0.86602540378444 - 0.37796447300923i, 1.00000000000000).
\end{aligned}$$

After reordering we have

$$\begin{aligned}
(\tilde{\alpha}_1, \tilde{\beta}_1) &= (2.97791477286351 + 1.29966855807374i, 3.43859979147302),\\
(\tilde{\alpha}_2, \tilde{\beta}_2) &= (-1.43573050214952 + 0.62660416225306i, -1.65783878379957),\\
(\tilde{\alpha}_3, \tilde{\beta}_3) &= (0.30383422966230, 0.17541877428455),
\end{aligned}$$

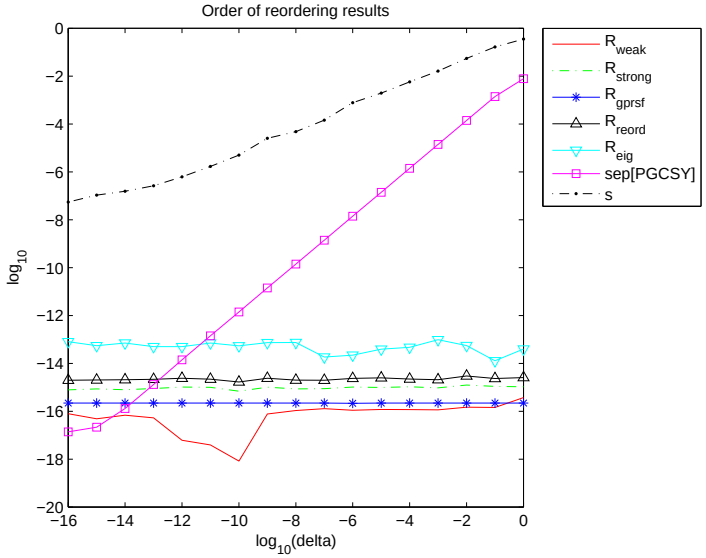

Figure 7.1: Results from reordering the eigenvalues of Examples III and IV with $\delta \in [0, 1]$. The displayed quantities are the same as in Tables 7.1-7.2. The horizontal axis shows the logarithm of the parameter δ and the vertical axis displays the logarithm of the computed quantities.

which define eigenvalues $\tilde{\lambda}_{1,2} = 0.86602540378444 \pm 0.37796447300923i$ and $\tilde{\lambda}_3 = 1.73205080756888$.

Example VI. Consider reordering the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \infty$. The original (α, β)-pairs are

$$\begin{aligned}(\alpha_1, \beta_1) &= (-0.9999999999999986, 1.000000000000000),\\ (\alpha_2, \beta_2) &= (1.000000000000000, 0.000000000000000).\end{aligned}$$

After reordering we have

$$\begin{aligned}(\tilde{\alpha}_1, \tilde{\beta}_1) &= (-1.564941642946474\text{E}{-5}, 0.000000000000000),\\ (\tilde{\alpha}_2, \tilde{\beta}_2) &= (6.390014634138052\text{E}{+4}, 6.390014634138062\text{E}{+4}),\end{aligned}$$

which correspond to the eigenvalues $\tilde{\lambda}_1 = -\infty$ and $\tilde{\lambda}_2 = 0.9999999999999985$.

8 Extensions of the K-cyclic reordering method

Below, we briefly describe two straightforward extensions to the direct reordering algorithm.

8.1 K-cyclic swapping in complex arithmetic

The algorithm described in this paper can easily be adopted to the complex case. Then we only have to cope with reordering of 1×1 blocks ($p_1 = p_2 = 1$)

using unitary matrices in the K-cyclic equivalence transformations. The main alternative is to reorder each adjacent eigenvalue pair by propagating a (Givens) rotation through the K-cyclic matrix pair [8], which requires more or less sophisticated implementation techniques to avoid underflow or overflow in the result [9], especially when the period K becomes large (see also Example 3 in [14]). Even with a careful implementation, the rotation approach may fail—for Example I we get the following swapped *complex* eigenvalues

$$\begin{aligned} \tilde{\lambda}_1 &= -1.87282049572853 + 0.58861866785157i, \\ \tilde{\lambda}_2 &= 1.74709648107590 + 0.47770138864644i. \end{aligned}$$

As can be seen from the results in the previous section, our algorithm achieves significantly higher accuracy for this particular example.

8.2 Reordering in even more general matrix products

Reordering can also be considered in matrix products of the form

$$A_{K-1}^{s_{K-1}} A_{K-2}^{s_{K-2}} \cdots A_0^{s_0}, \qquad s_0, \ldots, s_{K-1} \in \{1, -1\}, \tag{8.1}$$

which is needed, e.g., in [4]. This can be accomplished by insertion of identity matrices into the matrix pair sequence so that the exponent structure falls back on the same structure as in Equation (1.2), i.e., every second matrix is an inverse. By this procedure, the direct reordering method presented in the paper is applicable to any product of the general form (8.1). The obvious drawback is that the new periodic matrix pair sequence in the worst case will have two times as large period as the original pair.

We remark that an efficient implementation that is able to handle matrix products of the form (8.1) must rely on a data structure that allows insertion of new matrices into the sequence without to much data copying. These, and other implementation aspects, will be addressed in our forthcoming toolbox for periodic eigenvalue problems.

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